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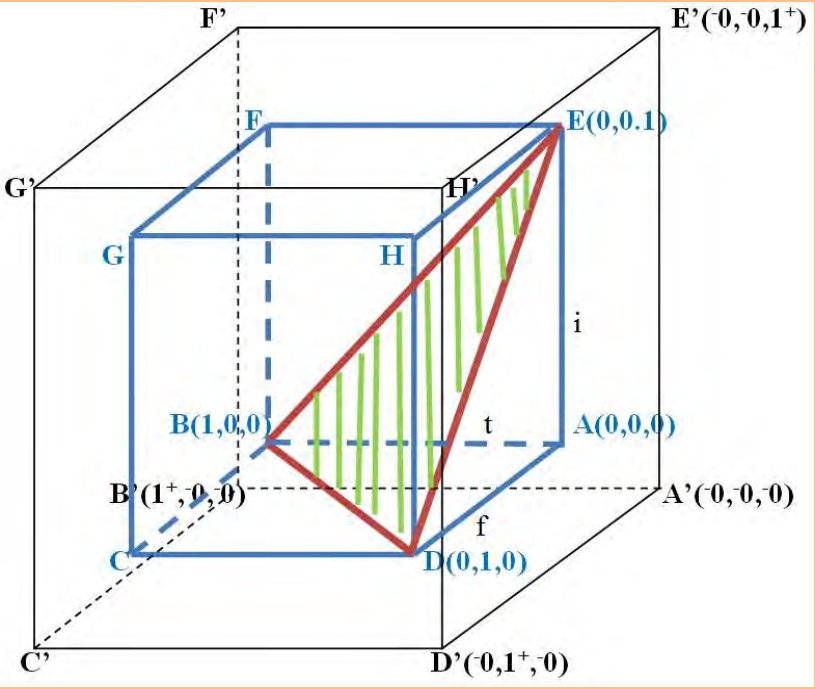

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Neutrosophic Sets and Systems

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Single Valued Neutrosophic Hyperbolic Sine Similarity Measure Based MADM Strategy

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Abstract: In this paper, we introduce new type of similarity measures for single valued neutrosophic sets based on hyperbolic sine function. The new similarity measures are namely, single valued neutrosophic hyperbolic sine similarity measure and weighted single valued neutrosophic hyperbolic sine similarity measure. We prove the basic properties of the proposed similarity measures. We also develop a multi-attribute decision-making strategy for single valued neutrosophic set based on the proposed weighted similarity measure. We present a numerical example to verify the practicability of the proposed strategy. Finally, we present a comparison of the proposed strategy with the existing strategies to exhibit the effectiveness and practicality of the proposed strategy.

Keywords: Single valued neutrosophic set, Hyperbolic sine function, Similarity measure, MADM, Compromise function

1 Introduction

Smarandache [1] introduced the concept of neutrosophic set (NS) to deal with imprecise and indeterminate data. In the concept of NS, truth-membership, indeterminacy-membership, and falsity-membership are independent. Indeterminacy plays an important role in many real world decision-making problems. NS generalizes the Cantor set discovered by Smith [2] in 1874 and introduced by German mathematician Cantor [3] in 1883, fuzzy set introduced by Zadeh [4], intuitionistic fuzzy set proposed by Atanassov [5]. Wang et al. [6] introduced the concept of single valued neutrosophic set (SVNS) that is the sub-class of a neutrosophic set. SVNS is capable to represent imprecise, incomplete, and inconsistent information that manifest the real world.

Neutrosophic sets and its various extensions have been studied and applied in different fields such as medical diagnosis [7, 8, 9], decision making problems [10, 11, 12, 13, 14], social problems [15, 16], educational problem [17, 18], conflict resolution [19], image processing [20, 21, 22], etc.

The concept of similarity is very important in studying almost every scientific field. Many strategies have been proposed for measuring the degree of similarity between fuzzy sets studied by Chen [23], Chen et al. [24], Hyung et al. [25], Pappis and Karacapilidis [26], Pramanik and Roy [27], etc. Several strategies have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets studied by Xu [28], Papakostas et al. [29], Biswas and Pramanik [30], Mondal and Pramanik [31], etc. However, these strategies are not capable of dealing with the similarity measures involving indeterminacy. SVNS can handle this situation. In the literature, few studies have addressed similarity measures for neutrosophic sets and single valued neutrosophic sets [32, 33, 34, 35].

Ye [36] proposed an MADM method with completely unknown weights based on similarity measures under SVNS environment. Ye [37] proposed vector similarity measures of simplified neutrosophic sets and applied it in multi-criteria decision making problems. Ye [38] developed improved cosine similarity measures of simplified neutrosophic sets for medical diagnosis. Ye [39] also proposed exponential similarity measure of neutrosophic numbers for fault diagnoses of steam turbine. Ye [40] developed clustering algorithms based on similarity measures for SVNSs. Ye and Ye [41] proposed Dice similarity measure between single valued neutrosophic multisets. Ye et al. [42] proposed distance-based similarity measures of single valued neutrosophic multisets for medical diagnosis. Ye and Fu [43] developed a single valued neutrosophic similarity measure based on tangent function for multi-period medical diagnosis.

In hybrid environment Pramanik and Mondal [44] proposed cosine similarity measure of rough neutrosophic sets and provided its application in medical diagnosis. Pramanik and Mondal [45] also proposed cotangent

similarity measure of rough neutrosophic sets and its application to medical diagnosis.

Research gap: MADM strategy using similarity measure based on hyperbolic sine function under single valued neutrosophic environment is yet to appear.

Research questions:

- Is it possible to define a new similarity measure between single valued neutrosophic sets using hyperbolic sine function?
- Is it possible to develop a new MADM strategy based on the proposed similarity measures in single valued neutrosophic environment?

Having motivated from the above researches on neutrosophic similarity measures, we have introduced the concept of hyperbolic sine similarity measure for SVNS environment. The new similarity measures called single valued neutrosophic hyperbolic sine similarity measure (SVNHSSM) and single valued neutrosophic weighted hyperbolic sine similarity measure (SVNWHSSM). The properties of hyperbolic sine similarity are established. We have developed a MADM model using the proposed SVNWHSSM. The proposed hyperbolic sine similarity measure is applied to multi-attribute decision making.

The objectives of the paper:

- To define hyperbolic sine similarity measures for SVNS environment and prove some of it's basic properties.
- To define conpromise function for determining unknown weight of attributes.
- To develop a multi-attribute decision making model based on proposed similarity measures.
- To present a numerical example for the efficiency and effectiveness of the proposed strategy.

Rest of the paper is structured as follows. Section 2 presents preliminaries of neutrosophic sets and single valued neutrosophic sets. Section 3 is devoted to introduce hyperbolic sine similarity measure for SVNSs and some of its properties. Section 4 presents a method to determine unknown attribute weights. Section 5 presents a novel decision making strategy based on proposed neutrosophic hyperbolic sine similarity measure. Section 6 presents an illustrative example for the application of the proposed method. Section 7 presents a comparison analysis for the applicability of the proposed strategy. Section 8 presents the main contributions of the proposed strategy. Finally, section 9 presents concluding remarks and scope of future research.

2 Neutrosophic preliminaries

2.1 Neutrosophic set (NS)

Definition 2.1 [1] Let U be a universe of discourse. Then the neutrosophic set P can be presented of the form:

$P = \{< x:T_P(x), I_P(x), F_P(x)> \mid x \in U\}$, where the functions $T, I, F: U \rightarrow {}^{-}]0,1^{+}[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in U$ to the set P satisfying the following the condition.

$$^{-}0 \leq \sup T_P(x) + \sup I_P(x) + \sup F_P(x) \leq 3^{+}$$

2.2 Single valued neutrosophic set (SVNS)

Definition 2.2 [6] Let X be a space of points with generic elements in X denoted by x. A SVNS P in X is characterized by a truth-membership function $T_P(x)$, an indeterminacy-membership function $I_P(x)$, and a falsity membership function $F_P(x)$, for each point x in X.

$T_P(x), I_P(x), F_P(x) \in [0, 1]$. When X is continuous, a SVNS P can be written as follows:

$$P = \int_X \frac{<T_P(x), I_P(x), F_P(x)>}{x} : x \in X$$

When X is discrete, a SVNS P can be written as follows:

$$P = \sum_{i=1}^{n} \frac{<T_P(x_i), I_P(x_i), F_P(x_i)>}{x_i} : x_i \in X$$

For two SVNSs,

$P_{SVNS} = \{<x: T_P(x), I_P(x), F_P(x)> \mid x \in X\}$ and
$Q_{SVNS} = \{<x, T_Q(x), I_Q(x), F_Q(x)> \mid x \in X\}$ the two relations are defined as follows:

(1) $P_{SVNS} \subseteq Q_{SVNS}$ if and only if $T_P(x) \leq T_Q(x)$, $I_P(x) \geq I_Q(x)$, $F_P(x) \geq F_Q(x)$

(2) $P_{SVNS} = Q_{SVNS}$ if and only if $T_P(x) = T_Q(x)$, $I_P(x) = I_Q(x)$, $F_P(x) = F_Q(x)$ for any $x \in X$.

3. Hyperbolic sine similarity measures for SVNSs

Let $A = <x(T_A(x), I_A(x), F_A(x))>$ and $B = <x(T_B(x), I_B(x), F_B(x))>$ be two SVNSs. Now hyperbolic sine similarity function which measures the similarity between two SVNSs can be presented as follows (see Eqn. 1):

$$\text{SVNHSSM}(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sinh\left(\begin{matrix}|T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)| \\ +|F_A(x_i)-F_B(x_i)|\end{matrix}\right)}{11}\right) \quad (1)$$

Theorem 1. The defined hyperbolic sine similarity measure SVNHSSM(A, B) between SVNSs A and B satisfies the following properties:

1. $0 \leq$ SVNHSSM(A, B) ≤ 1
2. SVNHSSM(A, B) = 1 if and only if $A = B$
3. SVNHSSM (A, B) = SVNHSSM(B, A)
4. If R is a SVNS in X and $A \subset B \subset R$ then SVNHSSM(A, R) $\leq$ SVNHSSM(A, B) and SVNHSSM(A, R) $\leq$ SVNHSSM(B, R).

Proofs:

1. For two neutrosophic sets A and B,

$$0 \leq T_A(x_i), I_A(x_i), F_A(x_i), T_B(x_i), I_B(x_i), F_B(x_i) \leq 1$$

$$\Rightarrow 0 \leq |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)| \leq 3$$

$$\Rightarrow 0 \leq \left(\frac{\sinh\left(\begin{matrix} |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| \\ + |F_A(x_i) - F_B(x_i)| \end{matrix} \right)}{11} \right) \leq 1$$

Hence $0 \leq$ SVNHSSM(A, B) $\leq$ 1

2. For any two SVNSs A and B, if $A = B$,

$$\Rightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$$

$$\Rightarrow |T_A(x) - T_B(x)| = 0 \ , \ |I_A(x) - I_B(x)| = 0 \ , \ |F_A(x) - F_B(x)| = 0$$

Hence SVNHSSM(A, B) = 1.

Conversely,

SVNHSSM(A, B) = 1

$$\Rightarrow |T_A(x) - T_B(x)| = 0 \ , \ |I_A(x) - I_B(x)| = 0 \ , \ |F_A(x) - F_B(x)| = 0 .$$

This implies, $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$.

Hence $A = B$.

3. Since,

$$|T_A(x) - T_B(x)| = |T_B(x) - T_A(x)| \ ,$$

$$|I_A(x) - I_B(x)| = |I_B(x) - I_A(x)| \ ,$$

$$|F_A(x) - F_B(x)| = |F_B(x) - F_A(x)| .$$

We can write, SVNHSSM(A, B) = SVNHSSM(B, A).

4. $A \subset B \subset R$

$$\Rightarrow T_A(x) \leq T_B(x) \leq T_R(x), I_A(x) \geq I_B(x) \geq I_R(x), F_A(x) \geq F_B(x) \geq F_R(x) \text{ for } x \in X.$$

Now we have the following inequalities:

$$|T_A(x) - T_B(x)| \leq |T_A(x) - T_R(x)| \ ,$$

$$|T_B(x) - T_R(x)| \leq |T_A(x) - T_R(x)| \ ;$$

$$|I_A(x) - I_B(x)| \leq |I_A(x) - I_R(x)| \ ,$$

$$|I_B(x) - I_R(x)| \leq |I_A(x) - I_R(x)| \ ;$$

$$|F_A(x) - F_B(x)| \leq |F_A(x) - F_R(x)| \ ,$$

$$|F_B(x) - F_R(x)| \leq |F_A(x) - F_R(x)| .$$

Thus, SVNHSSM(A, R) $\leq$ SVNHSSM(A, B) and SVNHSSM(A, R) $\leq$ SVNHSSM(B, R).

3.1 Weighted hyperbolic sine similarity measures for SVNSs

Let $A = \langle x(T_A(x), I_A(x), F_A(x)) \rangle$ and $B = \langle x(T_B(x), I_B(x), F_B(x)) \rangle$ be two SVNSs. Now weighted hyperbolic sine similarity function which measures the similarity between two SVNSs can be presented as follows (see Eqn. 2):

$$\text{SVN WHSSM } (A, B) = 1 - \sum_{i=1}^{n} w_i \left(\frac{\sinh\left(\begin{matrix} |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| \\ + |F_A(x_i) - F_B(x_i)| \end{matrix} \right)}{11} \right) \quad (2)$$

Here, $0 \leq w_i \leq 1$, $\sum_{i=1}^{n} w_i = 1$.

Theorem 2. The defined weighted hyperbolic sine similarity measure SVNWHSSM(A, B) between SVNSs A and B satisfies the following properties:

1. $0 \leq$ SVNWHSSM(A, B) ≤ 1
2. SVNWHSSM (A, B) = 1 if and only if $A = B$
3. SVNWHSSM (A, B) = SVNWHSSM (B, A)
4. If R is a SVNS in X and $A \subset B \subset R$ then SVNWHSSM (A, R) $\leq$ SVNWHSSM(A, B) and SVNWHSSM (A, R) $\leq$ SVNWHSSM (B, R).

Proofs:

1. For two neutrosophic sets A and B,

$$0 \leq T_A(x_i), I_A(x_i), F_A(x_i), T_B(x_i), I_B(x_i), F_B(x_i) \leq 1$$

$$\Rightarrow 0 \leq |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)| \leq 3$$

$$\Rightarrow 0 \leq \left(\frac{\sinh\left(\begin{matrix} |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| \\ + |F_A(x_i) - F_B(x_i)| \end{matrix} \right)}{11} \right) \leq 1$$

Again, $0 \leq w_i \leq 1$, $\sum_{i=1}^{n} w_i = 1$.

Hence $0 \leq$ SVNWHSSM(A, B) $\leq$ 1

2. For any two SVNSs A and B, if $A = B$,

$\Rightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$

$\Rightarrow |T_A(x) - T_B(x)| = 0$, $|I_A(x) - I_B(x)| = 0$,

$|F_A(x) - F_B(x)| = 0$

Hence SVNWHSSM(A, B) = 1.

Conversely,

SVNWHSSM(A, B) = 1

$\Rightarrow |T_A(x) - T_B(x)| = 0$, $|I_A(x) - I_B(x)| = 0$,

$|F_A(x) - F_B(x)| = 0$.

This implies, $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$.

Hence $A = B$.

3. Since,

$|T_A(x) - T_B(x)| = |T_B(x) - T_A(x)|$,

$|I_A(x) - I_B(x)| = |I_B(x) - I_A(x)|$,

$|F_A(x) - F_B(x)| = |F_B(x) - F_A(x)|$.

We can write, SVNWHSSM(A, B) = SVNWHSSM(B, A).

4. $A \subset B \subset R$

$\Rightarrow T_A(x) \le T_B(x) \le T_R(x), I_A(x) \ge I_B(x) \ge I_R(x)$,

$F_A(x) \ge F_B(x) \ge F_R(x)$ for $x \in X$.

Now we have the following inequalities:

$|T_A(x) - T_B(x)| \le |T_A(x) - T_R(x)|$,

$|T_B(x) - T_R(x)| \le |T_A(x) - T_R(x)|$;

$|I_A(x) - I_B(x)| \le |I_A(x) - I_R(x)|$,

$|I_B(x) - I_R(x)| \le |I_A(x) - I_R(x)|$;

$|F_A(x) - F_B(x)| \le |F_A(x) - F_R(x)|$,

$|F_B(x) - F_R(x)| \le |F_A(x) - F_R(x)|$.

Thus SVNWHSSM(A, R) $\le$ SVNWHSSM(A, B) and SVNWHSSM(A, R) $\le$ SVNWHSSM(B, R).

4. Determination of unknown attribute weights

When attribute weights are completely unknown to decision makers, the entropy measure [46] can be used to calculate attribute weights. Biswas et al. [47] employed entropy measure for MADM problems to determine completely unknown attribute weights of SVNSs.

4.1 Compromise function

The compromise function of a SVNS $A = \langle T_{ij}^A, I_{ij}^A, F_{ij}^A \rangle$ (i = 1, 2, ..., m; j = 1, 2, ..., n) is defined as follows (see Eqn. 3):

$$C_j(A) = \sum_{i=1}^{m} \left(2 + T_{ij}^A - I_{ij}^A - F_{ij}^A\right) / 3 \quad (3)$$

The weight of j-th attribute is defined as follows (see Eqn. 4).

$$w_j = \frac{C_j(A)}{\sum_{j=1}^{n} C_j(A)} \quad (4)$$

Here, $\sum_{j=1}^{n} w_j = 1$.

Theorem 3. The compromise function $C_j(A)$ satisfies the following properties:

P1. $C_j(A) = 1$, if $T_{ij} = 1, F_{ij} = I_{ij} = 0$.

P2. $C_j(A) = 0$, if $\langle T_{ij}, I_{ij}, F_{ij} \rangle = \langle 0, 1, 1 \rangle$.

P3. $C_j(A) \ge E_j(B)$, if $T_{ij}^A \ge T_{ij}^B$ and $I_{ij}^A + F_{ij}^A \le I_{ij}^B + F_{ij}^B$.

Proofs.

P1. $T_{ij} = 1, F_{ij} = I_{ij} = 0$

$$\Rightarrow C_j(A) = \frac{1}{m} \sum_{i=1}^{m} 3/3 = \frac{1}{m}.m = 1$$

P2. $\langle T_{ij}, I_{ij}, F_{ij} \rangle = \langle 0, 1, 1 \rangle$.

$$\Rightarrow C_j(A) = \frac{1}{m} \sum_{i=1}^{m} 0/3 = 0$$

P3. $C_j(A) - C_j(B)$

$$\Rightarrow \left\{ \frac{1}{m} \sum_{i=1}^{m} \left(2 + T_{ij}^A - I_{ij}^A - F_{ij}^A\right)/3 - \frac{1}{m} \sum_{i=1}^{m} \left(2 + T_{ij}^B - I_{ij}^B - F_{ij}^B\right)/3 \right\} > 0$$

$\Rightarrow C_j(A) - C_j(B) > 0$, Since, $T_{ij}^A > T_{ij}^B$ and $I_{ij}^A + F_{ij}^A < I_{ij}^B + F_{ij}^B$.

Hence, $C_j(A) \ge C_j(B)$.

5. Decision making procedure

Let A_1, A_2 , ..., A_m be a discrete set of alternatives, C_1, C_2, ..., C_n be the set of attributes of each alternative. The values associated with the alternatives A_i (i = 1, 2,..., m) against the attribute C_j (j = 1, 2, ..., n) for MADM problem is presented in a SVNS based decision matrix.

The steps of decision-making (see Figure 2) based on single valued neutrosophic weighted hyperbolic sine similarity measure (SVNWHSSM) are presented using the following steps.

Step 1: Determination of the relation between alternatives and attributes

The relation between alternatives A_i (i = 1, 2, ..., m) and the attribute C_j (j = 1, 2, ..., n) is presented in the Eqn. (5).

$$D[A|C]=\begin{pmatrix} & C_1 & C_2 & \cdots & C_n \\ A_1 & \langle T_{11}, I_{11}, F_{11}\rangle & \langle T_{12}, I_{12}, F_{12}\rangle & \cdots & \langle T_{1n}, I_{1n}, F_{1n}\rangle \\ A_2 & \langle T_{21}, I_{21}, F_{21}\rangle & \langle T_{22}, I_{22}, F_{22}\rangle & \cdots & \langle T_{2n}, I_{2n}, F_{2n}\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & \langle T_{m1}, I_{m1}, F_{1m1}\rangle & \langle T_{m2}, I_{m2}, F_{m2}\rangle & \cdots & \langle T_{mn}, I_{mn}, F_{mn}\rangle \end{pmatrix} \quad (5)$$

Here $\langle T_{ij}, I_{ij}, F_{ij}\rangle$ (i = 1, 2, ..., m; j = 1, 2, ..., n) be SVNS assessment value.

Step 2: Determine the weights of attributes

Using the Eqn. (3) and (4), decision-maker calculates the weight of the attribute C_j (j = 1, 2, …, n).

Step 3: Determine ideal solution

Generally, the evaluation attribute can be categorized into two types: benefit type attribute and cost type attribute. In the proposed decision-making method, an ideal alternative can be identified by using a maximum operator for the benefit type attributes and a minimum operator for the cost type attributes to determine the best value of each attribute among all the alternatives. Therefore, we define an ideal alternative as follows:

A^* = {C_1^*, C_2^*, … , C_m^*}.

Here, benefit attribute C_j^* can be presented as follows:

$$C_j^*=\left\lfloor \max_i T_{C_j}^{(A_i)}, \min_i I_{C_j}^{(A_i)}, \min_i F_{C_j}^{(A_i)} \right\rfloor \quad (6)$$

for j = 1, 2, ..., n.

Similarly, the cost attribute C_j^* can be presented as follows:

$$C_j^*=\left\lfloor \min_i T_{C_j}^{(A_i)}, \max_i I_{C_j}^{(A_i)}, \max_i F_{C_j}^{(A_i)} \right\rfloor \quad (7)$$

for j = 1, 2, ..., n

Step 4: Determine the similarity values

Using Eqns. (2) and (5), calculate SVNWHSSM values for each alternative between positive (or negative) ideal solutions and corresponding single valued neutrosophic from decision matrix $D[A|C]$.

Step 5: Ranking the alternatives

Ranking the alternatives is prepared based on the descending order of similarity measures. Highest value indicates the best alternative.

Step 6: End

6. Numerical example

In this section, we illustrate a numerical example as an application of the proposed approach. We consider a decision-making problem stated as follows. Suppose a person who wants to purchase a SIM card for his/her mobile connection. Therefore, it is necessary to select suitable SIM card for his/her mobile connection. After initial screening, there are four possible alternatives (SIM cards) for mobile connection. The alternatives (SIM cards) are presented as follows:

- A_1: Airtel
- A_2: Vodafone
- A_3: BSNL
- A_4: Reliance Jio

The person must take a decision based on the following five attributes of SIM cards:

- C_1: Service quality
- C_2: Cost
- C_3: Initial talk time
- C_4: Call rate per second
- C_5: Internet and other facilities

The decision-making strategy is presented using the following steps.

Step 1: Determine the relation between alternatives and attributes

The relation between alternatives A_1, A_2, A_3, and A_4 and the attributes C_1, C_2, C_3, C_4, C_5 is presented in the Eqn. (8).

$$D[A|C_1,C_2,C_3,C_4,C_5]=\begin{pmatrix} & C_1 & C_2 & C_3 & C_4 & C_5 \\ A_1 & \langle .7, .3, .3\rangle & \langle .6, .4, .3\rangle & \langle .8, .1, .1\rangle & \langle .5, .4, .4\rangle & \langle .5, .3, .2\rangle \\ A_2 & \langle .5, .3, .1\rangle & \langle .7, .1, .3\rangle & \langle .7, .3, .1\rangle & \langle .6, .1, .1\rangle & \langle .5, .2, .3\rangle \\ A_3 & \langle .8, .2, .2\rangle & \langle .6, .4, .3\rangle & \langle .6, 0, .1\rangle & \langle .7, .3, 0\rangle & \langle .5, .3, .4\rangle \\ A_4 & \langle .6, .1, .3\rangle & \langle .5, .1, .2\rangle & \langle .6, .3, .1\rangle & \langle .5, .1, .2\rangle & \langle .9, .1, .1\rangle \end{pmatrix} \quad (8)$$

Step 2: Determine the weights of attributes

Using the Eq. (3) and (4), we calculate the weight of the attributes C_1, C_2, C_3, C_4, C_5 as follows:

[w_1, w_2, w_3, w_4, w_5] =

[0.2023, 0.1917, 0.2078, 0.2009, 0.1973]

Step 3: Determine ideal solution

In this problem, attributes C_1, C_3, C_4, C_5 are benefit type attributes and , C_2 is the cost type attribute.

A^* = {(0.8, 0.1, 0.1), (0.5, 0.4, 0.3), (0.8, 0.0, 0.1), (0.7, 0.1, 0.0), (0.9, 0.1, 0.1)}.

Step 4: Determine the weighted similarity values

Using Eq. (2) and Eq. (8), we calculate similarity measure values for each alternative as follows.

SVNWHSSM(A^*, A_1) = 0.92422

SVNWHSSM(A^*, A_2) = 0.95629

SVNWHSSM(A^*, A_3) = 0.97866

SVNWHSSM(A^*, A_4) = 0.96795

Step 5: Ranking the alternatives

Ranking the alternatives is prepared based on the descending order of similarity measures (see Figure 1). Now the final ranking order will be as follows.

$A_3 \succ A_4 \succ A_2 \succ A_1$

Highest value indicates the best alternative.

Step 6: End

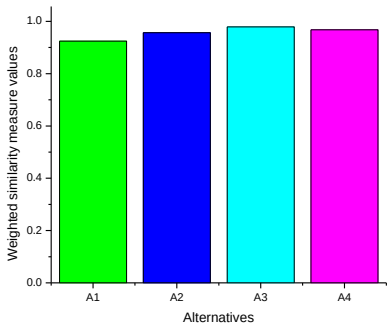

FIGURE 1: Graphical representation of alternatives versus weighted similarity measures.

7. Comparison analysis

The ranking results calculated from proposed strategy and different existing strategies [38, 48, 49, 50] are furnished in Table 1. We observe that the ranking results obtained from proposed and existing strategies in the literature differ.

The proposed strategy reflects that the optimal alternative is A_3. The ranking result obtained from Ye [38] is similar to the proposed strategy. The ranking results obtained from Ye and Zhang [48] and Mondal and Pramanik [49] differ from the optimal result of the proposed strategy. In Ye [50], the ranking order differs but the best alternative is the same to the proposed strategy.

Table 1 The ranking results of existing strategies

Strategies	Ranking results
Ye and Zhang[48]	$A_4 \succ A_2 \succ A_3 \succ A_1$
Mondal and Pramanik [49]	$A_4 \succ A_3 \succ A_2 \succ A_1$
Ye [38]	$A_3 \succ A_4 \succ A_2 \succ A_1$
Ye [50]	$A_3 \succ A_2 \succ A_4 \succ A_1$
Proposed strategy	$A_3 \succ A_4 \succ A_2 \succ A_1$

8. Contributions of the proposed strategy

1) SVNHSSM and SVNWHSSM in SVNS environment are firstly defined in the literature. We have also proved their basic properties.
2) We have proposed 'compromise function' for calculating unknown weights structure of attributes in SVNS environment.
3) We develop a decision making strategy based on the proposed weighted similarity measure (SVNWHSSM).
4) Steps and calculations of the proposed strategy are easy to use.
5) We have solved a numerical example to show the feasibility, applicability, and effectiveness of the proposed strategy.

9. Conclusion

In the paper, we have proposed hyperbolic sine similarity measure and weighted hyperbolic sine similarity measures for SVNSs and proved their basic properties. We have proposed compromise function to determine unknown weights of the attributes in SVNS environment. We have developed a novel MADM strategy based on the proposed weighted similarity measure to solve decision problems. We have solved a numerical problem and compared the obtained result with other existing strategies to demonstrate the effectiveness of the proposed MADM strategy. The proposed MADM strategy can be applied in other decision-making problem such as supplier selection, pattern recognition, cluster analysis, medical diagnosis, weaver selection [51-53], fault diagnosis [54], brick selection [55-56], data mining [57], logistic centre location selection [58-60], teacher selection [61, 62], etc.

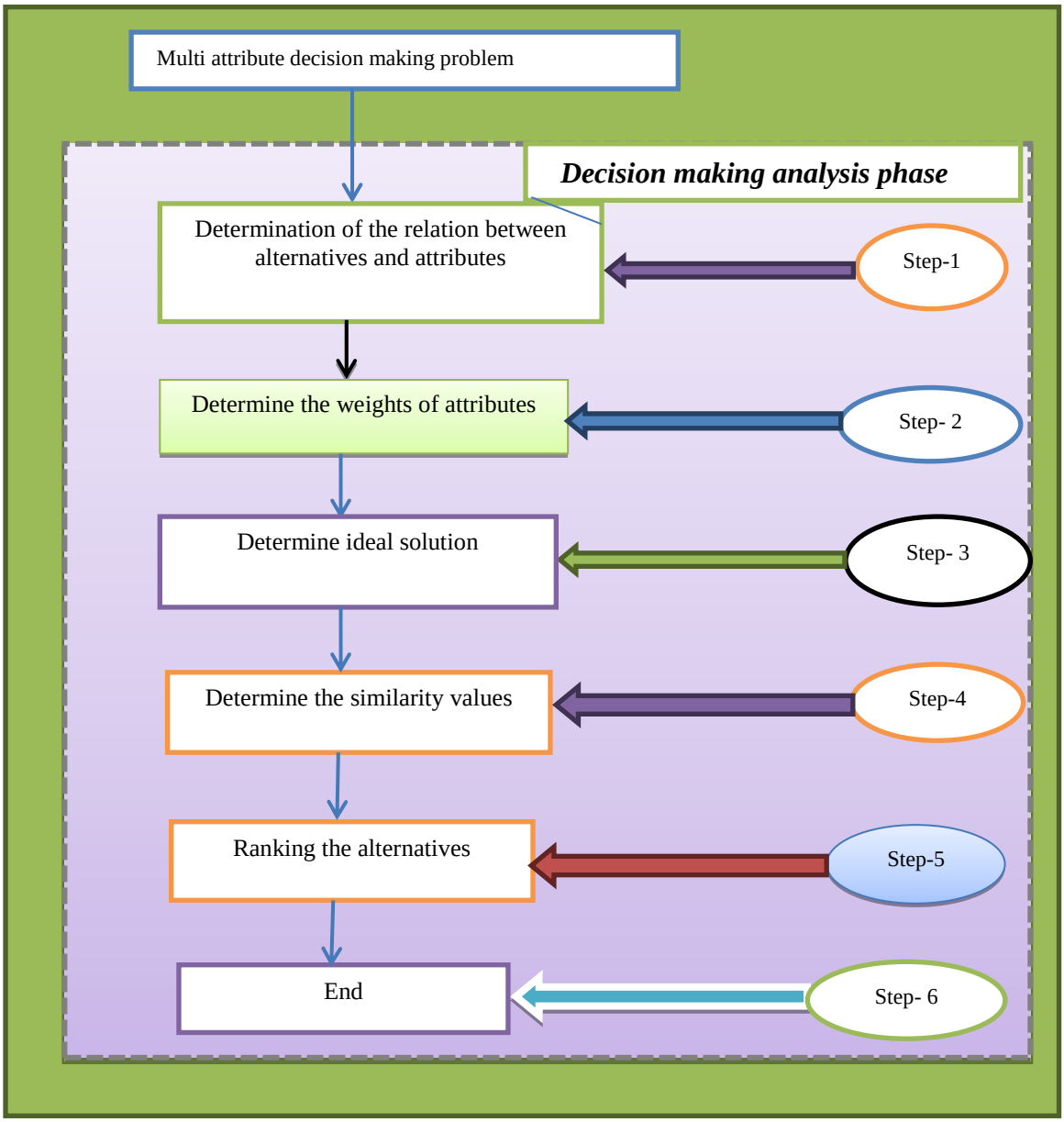

FIGURE 2: Phase diagram of the proposed decision making strategy

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Hybrid Binary Logarithm Similarity Measure for MAGDM Problems under SVNS Assessments

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Abstract: Single valued neutrosophic set is an important mathematical tool for tackling uncertainty in scientific and engineering problems because it can handle situation involving indeterminacy. In this research, we introduce new similarity measures for single valued neutrosophic sets based on binary logarithm function. We define two type of binary logarithm similarity measures and weighted binary logarithm similarity measures for single valued neutrosophic sets. Then we define hybrid binary logarithm similarity measure and weighted hybrid binary logarithm similarity measure for single valued neutrosophic sets. We prove the basic properties of the proposed measures. Then, we define a new entropy function for determining unknown attribute weights. We develop a novel multi attribute group decision making strategy for single valued neutrosophic sets based on the weighted hybrid binary logarithm similarity measure. We present an illustrative example to demonstrate the effectiveness of the proposed strategy. We conduct a sensitivity analysis of the developed strategy. We also present a comparison analysis between the obtained results from proposed strategy and different existing strategies in the literature.

Keywords: single valued neutrosophic set; binary logarithm function; similarity measure; entropy function; ideal solution; MAGDM

1 Introduction

Smarandache [1] introduced neutrosophic sets (NSs) to pave the way to deal with problems involving uncertainty, indeterminacy and inconsistency. Wang et al. [2] grounded the concept of single valued neutrosophic sets (SVNSs), a subclass of NSs to tackle engineering and scientific problems. SVNSs have been applied to solve various problems in different fields such as medical problems [3–5], decision making problems [6–18], conflict resolution [19], social problems [20–21] engineering problems [22-23], image processing problems [24–26] and so on.

The concept of similarity measure is very significant in studying almost every practical field. In the literature, few studies have addressed similarity measures for SNVSs [27–30]. Peng et al. [31] developed SVNSs based multi attribute decision making (MADM) strategy employing MABAC (Multi-Attributive Border Approximation area Comparison and similarity measure), TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) and a new similarity measure.

Ye [32] proposed cosine similarity measure based neutrosophic multiple attribute decision making (MADM) strategy. In order to overcome some disadvantages in the definition of cosine similarity measure, Ye [33] proposed 'improved cosine similarity measures' based on cosine function. Biswas et al. [34] studied cosine similarity measure based MCDM with trapezoidal fuzzy neutrosophic numbers. Pramanik and Mondal [35] proposed weighted fuzzy similarity measure based on tangent function. Mondal and Pramanik [36] proposed intuitionistic fuzzy similarity measure based on tangent function. Mondal and Pramanik [37] developed tangent similarity measure of SVNSs and applied it to MADM. Ye and Fu [38] studied medical diagnosis problem using a SVNSs similarity measure based on tangent function. Can and Ozguven [39] studied a MADM problem for adjusting the proportional-integral-derivative (PID) coefficients based on neutrosophic Hamming, Euclidean, set-theoretic, Dice, and Jaccard similarity measures.

Several studies [40–42] have been reported in the literature for multi-attribute group decision making (MAGDM) in neutrosophic environment. Ye [43] studied the similarity measure based on distance function of SVNSs and applied it to MAGDM. Ye [44] developed several clustering methods using distance-based similarity measures for SVNSs.

Mondal et al. [45] proposed sine hyperbolic similarity measure for solving MADM problems. Mondal et al. [46] also proposed tangent similarity measure to deal with MADM problems for interval neutrosophic environment.

Lu and Ye [47] proposed logarithmic similarity measure for interval valued fuzzy set [48] and applied it in fault diagnosis strategy.

Research gap:

MAGDM strategy using similarity measure based on binary logarithm function under single valued neutrosophic environment is yet to appear.

Research questions:

- Is it possible to define a new similarity measure between single valued neutrosophic sets using binary logarithm function?
- Is it possible to define a new entropy function for single valued neutrosophic sets for determining unknown attribute weights?
- Is it possible to develop a new MAGDM strategy based on the proposed similarity measures in single valued neutrosophic environment?

The objectives of the paper:

- To define binary logarithm similarity measures for SVNS environment and prove the basic properties.
- To define a new entropy function for determining unknown weight of attributes.
- To develop a multi-attribute droup decision making model based on proposed similarity measures.
- To present a numerical example for the efficiency and effectiveness of the proposed strategy.

Having motivated from the above researches on neutrosophic similarity measures, we introduce the concept of binary logarithm similarity measures for SVNS environment. The properties of binary logarithm similarity measures are established. We also propose a new entropy function to determine unknown attribute weights. We develope a MAGDM strategy using the proposed hybrid binary logarithm similarity measures. The proposed similarity measure is applied to a MAGDM problem.

The structure of the paper is as follows. Section 2 presents basic concepts of NSs, operations on NSs, SVNSs and operations on SVNSs. Section 3 proposes binary logarithm similarity measures and weighted binary logarithm similarity measures, hybrid binary logarithm similarity measure (HBLSM), weighted hybrid binary logarithm similarity measure (WHBLSM) in SVNSs environment. Section 4 proposes a new entropy measure to calculate unknown attribute weights and proves basic properties of entropy function. Section 5 presents a MAGDM strategy based weighted hybrid binary logarithm similarity measure. Section 6 presents an illustrative example to demonstrate the applicability and feasibility of the proposed strategies. Section 7 presents a sensitivity analysis for the results of the numerical example. Section 8 conducts a comparative analysis with the other existing strategies. Section 9 presents the key contribution of the paper. Section 10 summarizes the paper and discusses future scope of research.

2 Preliminaries

In this section, the concepts of NSs, SVNSs, operations on NSs and SVNSs and binary logarithm function are outlined.

2.1 Neutrosophic set (NS)

Assume that X be an universe of discourse. Then a neutrosophic sets [1] N can be defined as follows:

$$N = \{< x: T_N(x), I_N(x), F_N(x) > \mid x \in X\}.$$

Here the functions T, I and F define respectively the membership degree, the indeterminacy degree, and the non-membership degree of the element $x \in X$ to the set N. The three functions T, I and F satisfy the following the conditions:

- $T, I, F: X \rightarrow {}^{-}]0,1^{+}[$
- ${}^{-}0 \leq \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \leq 3^{+}$

For two neutrosophic sets $M = \{< x: T_M(x), I_M(x), F_M(x) > \mid x \in X\}$ and $N = \{< x, T_N(x), I_N(x), F_N(x) > \mid x \in X\}$, the two relations are defined as follows:

- $M \subseteq N$ if and only if $T_M(x) \leq T_N(x)$, $I_M(x) \geq I_N(x)$, $F_M(x) \geq F_N(x)$
- $M = N$ if and only if $T_M(x) = T_N(x)$, $I_M(x) = I_N(x)$, $F_M(x) = F_N(x)$.

2.2. Single valued Neutrosophic sets (SVNSs)

Assume that X be an universe of discourse. A SVNS [2] P in X is formed by a truth-membership function $T_P(x)$, an indeterminacy membership function $I_P(x)$, and a falsity membership function $F_P(x)$. For each point x in X, $T_P(x)$, $I_P(x)$, and $F_P(x) \in [0, 1]$.

For continuous case, a SVNS P can be expressed as follows:

$$P = \int_x \frac{< T_P(x), I_P(x), F_P(x) >}{x} : x \in X \ ,$$

For discrete case, a SVNS P can be expressed as follows:

$$P = \sum_{i=1}^{n} \frac{<T_P(x_i), I_P(x_i), F_P(x_i)>}{x_i} : x_i \in X$$

For two SVNSs $P = \{< x: T_P(x), I_P(x), F_P(x)> \mid x \in X\}$ and $Q = \{< x: T_Q(x), I_Q(x), F_Q(x)> \mid x \in X\}$, some definitions are stated below:

- $P \subseteq Q$ if and only if $T_P(x) \le T_Q(x)$, $I_P(x) \ge I_Q(x)$, and $F_P(x) \ge F_Q(x)$.
- $P \supseteq Q$ if and only if $T_P(x) \ge T_Q(x)$, $I_P(x) \le I_Q(x)$, and $F_P(x) \le F_Q(x)$.
- $P = Q$ if and only if $T_P(x) = T_Q(x)$, $I_P(x) = I_Q(x)$, and $F_P(x) = F_Q(x)$ for any $x \in X$.
- Complement of P i.e. $P^c = \{< x: F_P(x), 1- I_P(x), T_P(x)> \mid x \in X\}$.

2.3. Some arithmetic operations on SVNSs

Definition 1 [49]

Let $P = \langle T_P(x), I_P(x), F_P(x)\rangle$ and $Q = \langle T_Q(x), I_Q(x), F_Q(x)\rangle$ be any two SVNSs in a universe of discourse then arithmetic operations are stated as follows.

- $P \oplus Q = \begin{pmatrix} T_P(x) + T_Q(x) - T_P(x)T_Q(x), I_P(x)I_Q(x), \\ F_P(x)F_Q(x) \end{pmatrix}$
- $P \otimes Q = \begin{pmatrix} T_P(x)T_Q(x), I_P(x) + I_Q(x) - I_P(x)I_Q(x), \\ F_P(x) + F_Q(x) - F_P(x)F_Q(x) \end{pmatrix}$
- $\alpha P = \left(1-\left(1-T_P(x)^{\alpha}\right), (I_P(x))^{\alpha}, (F_P(x))^{\alpha}\right); \alpha > 0$
- $(P)^{\alpha} = \left((T_P(x))^{\alpha}, 1-\left(1-I_P(x)^{\alpha}\right), 1-\left(1-F_P(x)^{\alpha}\right)\right); \alpha > 0$

2.4. Binary logarithm function

In mathematics, the logarithm of the form $\log_2^x$, $x > 0$ is called binary logarithm function [50]. For example, the binary logarithm of 1 is 0, the binary logarithm of 4 is 2, the binary logarithm of 16 is 4, and the binary logarithm of 64 is 6.

3. Binary logarithm similarity measures for SVNSs

In this section, we define two types of binary logarithm similarity measures and their hybrid and weighted hybrid similarity measures.

3.1. Binary logarithm similarity measures of SVNSs (type-I)

Definition 2. Let $A = <x(T_A(x_i), I_P(x_i), F_P(x_i))>$ and $B = <x(T_B(x_i), I_B(x_i), F_B(x_i))>$ be any two SVNSs. The binary logarithm similarity measure (type-I) between SVNSs A and B are defined as follows:

$$BL_1(A,B) =$$

$$\frac{1}{n}\sum_{i=1}^{n} \log_2\left(2-\left(\frac{1}{3}\begin{pmatrix}|T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)| \\ +|F_A(x_i)-F_B(x_i)|\end{pmatrix}\right)\right) \quad (1)$$

Theorem 1. The binary logarithm similarity measure $BL_1(A,B)$ between any two SVNSs A and B satisfy the following properties:

P1. $0 \le BL_1(A,B) \le 1$

P2. $BL_1(A,B) = 1$, if and only if $A = B$

P3. $BL_1(A,B) = BL_1(B,A)$

P4. If C is a SVNS in X and $A \subseteq B \subseteq C$ then $BL_1(A,C) \le BL_1(A,B)$ and $BL_1(A,C) \le BL_1(B,C)$.

Proof 1.

From the definition of SVNS, we write,
$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ and
$0 \le T_B(x) + I_B(x) + F_B(x) \le 3$
$\Rightarrow$
$$0 \le |T_A(x_i)-T_B(x_i)| + |I_A(x_i)-I_B(x_i)| + |F_A(x_i)-F_B(x_i)| \le 3,$$

$$0 \le \max\begin{pmatrix}|T_A(x_i)-T_B(x_i)|, |I_A(x_i)-I_B(x_i)|, \\ |F_A(x_i)-F_B(x_i)|\end{pmatrix} \le 1$$

$\Rightarrow 0 \le BL_1(A,B) \le 1$.

Proof 2.

For any two SVNSs A and B,
$A = B$
$\Rightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$
$\Rightarrow |T_A(x)-T_B(x)| = 0, |I_A(x)-I_B(x)| = 0,$
$|F_A(x)-F_B(x)| = 0$
$\Rightarrow BL_1(A,B) = 1$.

Conversely,

for $BL_1(A,B) = 1$, we have,
$\Rightarrow |T_A(x)-T_B(x)| = 0, |I_A(x)-I_B(x)| = 0,$

$|F_A(x) - F_B(x)| = 0$

$\Rightarrow T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$

$\Rightarrow A = B$.

Proof 3.

We have,

$|T_A(x) - T_B(x)| = |T_B(x) - T_A(x)|$,

$|I_A(x) - I_B(x)| = |I_B(x) - I_A(x)|$,

$|F_A(x) - F_B(x)| = |F_B(x) - F_A(x)|$

$\Rightarrow \mathrm{BL}_1(A, B) = \mathrm{BL}_1(B, A)$.

Proof 4.

For $A \subseteq B \subseteq C$, we have,

$T_A(x) \le T_B(x) \le T_C(x)$, $I_A(x) \ge I_B(x) \ge I_C(x)$,

$F_A(x) \ge F_B(x) \ge F_C(x)$ for $x \in X$.

$\Rightarrow |T_A(x) - T_B(x)| \le |T_A(x) - T_C(x)|$,

$|T_B(x) - T_C(x)| \le |T_A(x) - T_C(x)|$;

$|I_A(x) - I_B(x)| \le |I_A(x) - I_C(x)|$,

$|I_B(x) - I_C(x)| \le |I_A(x) - I_C(x)|$;

$|F_A(x) - F_B(x)| \le |F_A(x) - F_C(x)|$,

$|F_B(x) - F_C(x)| \le |F_A(x) - F_C(x)|$.

$\Rightarrow \mathrm{BL}_1(A, C) \le \mathrm{BL}_1(A, B)$ and $\mathrm{BL}_1(A, C) \le \mathrm{BL}_1(B, C)$.

3.2. Binary logarithm similarity measures of SVNSs (type-II)

Definition 3. [51] Let $A = \langle x(T_A(x_i), I_P(x_i), F_P(x_i))\rangle$ and $B = \langle x(T_B(x_i), I_B(x_i), F_B(x_i))\rangle$ be any two SVNSs. The binary logarithm similarity measure (type-II) between SVNSs A and B are defined as follows:

$$\mathrm{BL}_2(A, B) = \frac{1}{n}\sum_{i=1}^{n} \log_2\left(2 - \max\left(\begin{matrix}|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, \\ |F_A(x_i) - F_B(x_i)|\end{matrix}\right)\right) \quad (2)$$

Theorem 2. The binary logarithm similarity measure $\mathrm{BL}_2(A, B)$ between any two SVNSs A and B satisfy the following properties:

P1. $0 \le \mathrm{BL}_2(A, B) \le 1$

P2. $\mathrm{BL}_2(A, B) = 1$, if and only if $A = B$

P3. $\mathrm{BL}_2(A, B) = \mathrm{BL}_2(B, A)$

P4. If C is a SVNS in X and $A \subseteq B \subseteq C$ then $\mathrm{BL}_2(A, C) \le \mathrm{BL}_2(A, B)$ and $\mathrm{BL}_2(A, C) \le \mathrm{BL}_2(B, C)$.

Proof.

Proofs of the properties are shown in [51].

3.3. Weighted binary logarithm similarity measures of SVNSs for type-I

Definition 4. Let $A = \langle x(T_A(x_i), I_P(x_i), F_P(x_i))\rangle$ and $B = \langle x(T_B(x_i), I_B(x_i), F_B(x_i))\rangle$ be any two SVNSs. Then the weighted binary logarithm similarity measure for type-I between SVNSs A and B are defined as follows:

$$\mathrm{BL}_1^w(A, B) = \sum_{i=1}^{n} w_i \log_2\left(2 - \left(\frac{1}{3}\left(\begin{matrix}|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| \\ + |F_A(x_i) - F_B(x_i)|\end{matrix}\right)\right)\right) \quad (3)$$

Here, $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$.

Theorem 3. The weighted binary logarithm similarity measures $\mathrm{BL}_1^w(A, B)$ between SVNSs A and B satisfy the following properties:

P1. $0 \le \mathrm{BL}_1^w(A, B) \le 1$

P2. $\mathrm{BL}_1^w(A, B) = 1$, if and only if $A = B$

P3. $\mathrm{BL}_1^w(A, B) = \mathrm{BL}_1^w(B, A)$

P4. If C is a SVNS in X and $A \subseteq B \subseteq C$, then $\mathrm{BL}_1^w(A, C) \le \mathrm{BL}_1^w(A, B)$ and $\mathrm{BL}_1^w(A, C) \le \mathrm{BL}_1^w(B, C)$;

$\sum_{i=1}^{n} w_i = 1$.

Proof 1.

From the definition of SVNSs A and B, we write,

$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ and

$0 \le T_B(x) + I_B(x) + F_B(x) \le 3$

$$\Rightarrow 0 \le \max\left(\begin{matrix}|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, \\ |F_A(x_i) - F_B(x_i)|\end{matrix}\right) \le 1$$

$\Rightarrow 0 \le |T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)| \le 3$,

$\Rightarrow 0 \le \mathrm{BL}_1^w(A, B) \le 1$. since, $\sum_{i=1}^{n} w_i = 1$.

Proof 2.

For any two SVNSs A and B if $A = B$, then we have,

$T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$

$\Rightarrow |T_A(x) - T_B(x)| = 0$, $|I_A(x) - I_B(x)| = 0$,

$|F_A(x) - F_B(x)| = 0$

$\Rightarrow \mathrm{BL}_1^w(A,B)=1$, (t = 1, 2), since $\sum_{i=1}^{n} w_i=1$.

Conversely,

For $\mathrm{BL}_1^w(A,B)=1$, then we have,

$\Rightarrow |T_A(x)-T_B(x)|=0$, $|I_A(x)-I_B(x)|=0$,

$|F_A(x)-F_B(x)|=0$

$\Rightarrow T_A(x)=T_B(x)$, $I_A(x)=I_B(x)$, $F_A(x)=F_B(x)$

$\Rightarrow A=B$, since $\sum_{i=1}^{n} w_i=1$.

Proof 3.

For any two SVNSs A and B, we have,

$|T_A(x)-T_B(x)|=|T_B(x)-T_A(x)|$,

$|I_A(x)-I_B(x)|=|I_B(x)-I_A(x)|$,

$|F_A(x)-F_B(x)|=|F_B(x)-F_A(x)|$

$\Rightarrow \mathrm{BL}_1^w(A,B)=\mathrm{BL}_1^w(B,A)$ for.

Proof 4.

For $A\subseteq B\subseteq C$, we have,

$T_A(x)\le T_B(x)\le T_C(x)$, $I_A(x)\ge I_B(x)\ge I_C(x)$,

$F_A(x)\ge F_B(x)\ge F_C(x)$ for $x\in X$.

$\Rightarrow |T_A(x)-T_B(x)|\le |T_A(x)-T_C(x)|$,

$|T_B(x)-T_C(x)|\le |T_A(x)-T_C(x)|$;

$|I_A(x)-I_B(x)|\le |I_A(x)-I_C(x)|$,

$|I_B(x)-I_C(x)|\le |I_A(x)-I_C(x)|$;

$|F_A(x)-F_B(x)|\le |F_A(x)-F_C(x)|$,

$|F_B(x)-F_C(x)|\le |F_A(x)-F_C(x)|$.

$\Rightarrow \mathrm{BL}_1^w(A,C)\le \mathrm{BL}_1^w(A,B)$ and $\mathrm{BL}_1^w(A,C)\le \mathrm{BL}_1^w(B,C)$ since $\sum_{i=1}^{n} w_i=1$.

3.4. Weighted binary logarithm similarity measures of SVNSs for type-II

Definition 5. [51] Let $A=\langle x(T_A(x_i), I_P(x_i), F_P(x_i))\rangle$ and $B=\langle x(T_B(x_i), I_B(x_i), F_B(x_i))\rangle$ be any two SVNSs. Then the weighted binary logarithm similarity measure (type-II between SVNSs A and B is defined as follows:

$\mathrm{BL}_2^w(A,B)=$

$$\sum_{i=1}^{n} w_i \log_2\left(2-\max\begin{pmatrix}|T_A(x_i)-T_B(x_i)|, |I_A(x_i)-I_B(x_i)|,\\ |F_A(x_i)-F_B(x_i)|\end{pmatrix}\right) \quad (4)$$

Here, $0\le w_i\le 1$ and $\sum_{i=1}^{n} w_i=1$.

Proof.

For proof, see [51].

3.3. Hybrid binary logarithm similarity measures (HBLSM) for SVNSs

Definition 6. Let $A=\langle x(T_A(x_i), I_P(x_i), F_P(x_i))\rangle$ and $B=\langle x(T_B(x_i), I_B(x_i), F_B(x_i))\rangle$ be any two SVNSs. The hybrid binary logarithm similarity measure between SVNSs A and B is defined as follows:

$\mathrm{BL}_{Hyb}(A,B)=$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\left\{\sum_{i=1}^{n}\log_2\left(2-\left(\frac{1}{3}\begin{pmatrix}|T_A(x_i)-T_B(x_i)|\\ +|I_A(x_i)-I_B(x_i)|\\ +|F_A(x_i)-F_B(x_i)|\end{pmatrix}\right)\right)\right\}\\ +(1-\lambda)\sum_{i=1}^{n}\log_2\left(2-\max\begin{pmatrix}|T_A(x_i)-T_B(x_i)|,\\ |I_A(x_i)-I_B(x_i)|,\\ |F_A(x_i)-F_B(x_i)|\end{pmatrix}\right)\end{array}\right] \quad (5)$$

Here, $0\le\lambda\le 1$.

Theorem 4. The hybrid binary logarithm similarity measure $\mathrm{BL}_{Hyb}(A,B)$ between any two SVNSs A and B satisfy the following properties:

P1. $0\le \mathrm{BL}_{Hyb}(A,B)\le 1$

P2. $\mathrm{BL}_{Hyb}(A,B)=1$, if and only if $A=B$

P3. $\mathrm{BL}_{Hyb}(A,B)=\mathrm{BL}_{Hyb}(B,A)$

P4. If C is a SVNS in X and $A\subseteq B\subseteq C$ then

$\mathrm{BL}_{Hyb}(A,C)\le \mathrm{BL}_{Hyb}(A,B)$

and $\mathrm{BL}_{Hyb}(A,C)\le \mathrm{BL}_{Hyb}(B,C)$.

Proof 1.

From the definition of SVNS, we write,

$0\le T_A(x)+I_A(x)+F_A(x)\le 3$ and

$0\le T_B(x)+I_B(x)+F_B(x)\le 3$

$$\Rightarrow 0 \le \max\left(\begin{matrix}|T_A(x_i)-T_B(x_i)|,|I_A(x_i)-I_B(x_i)|,\\ |F_A(x_i)-F_B(x_i)|\end{matrix}\right) \le 1$$

$$\Rightarrow \begin{matrix}0 \le |T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)|\\ +|F_A(x_i)-F_B(x_i)| \le 3\end{matrix};$$

$$\Rightarrow 0 \le \mathrm{BL}_{Hyb}(A,B) \le 1.$$

Proof 2.

For any two SVNSs A and B,
for $A = B$, we have,
$\Rightarrow T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$
$\Rightarrow |T_A(x)-T_B(x)| = 0$, $|I_A(x)-I_B(x)| = 0$,
$|F_A(x)-F_B(x)| = 0$
$\Rightarrow \mathrm{BL}_{Hyb}(A,B) = 1$.

Conversely,

for $\mathrm{BL}_{Hyb}(A,B) = 1$, we have,
$|T_A(x)-T_B(x)| = 0$, $|I_A(x)-I_B(x)| = 0$,
$|F_A(x)-F_B(x)| = 0$
$\Rightarrow T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$
$\Rightarrow A = B$.

Proof 3.

For any two SVNSs A and B, we have,
$|T_A(x)-T_B(x)| = |T_B(x)-T_A(x)|$,
$|I_A(x)-I_B(x)| = |I_B(x)-I_A(x)|$,
$|F_A(x)-F_B(x)| = |F_B(x)-F_A(x)|$
$\Rightarrow \mathrm{BL}_{Hyb}(A,B) = \mathrm{BL}_{Hyb}(B,A)$.

Proof 4.

For $A \subseteq B \subseteq C$, we have,
$T_A(x) \le T_B(x) \le T_C(x)$, $I_A(x) \ge I_B(x) \ge I_C(x)$,
$F_A(x) \ge F_B(x) \ge F_C(x)$ for $x \in X$.
$\Rightarrow |T_A(x)-T_B(x)| \le |T_A(x)-T_C(x)|$,
$|T_B(x)-T_C(x)| \le |T_A(x)-T_C(x)|$;
$|I_A(x)-I_B(x)| \le |I_A(x)-I_C(x)|$,
$|I_B(x)-I_C(x)| \le |I_A(x)-I_C(x)|$;
$|F_A(x)-F_B(x)| \le |F_A(x)-F_C(x)|$,
$|F_B(x)-F_C(x)| \le |F_A(x)-F_C(x)|$.
$\Rightarrow \mathrm{BL}_{Hyb}(A,C) \le \mathrm{BL}_{Hyb}(A,B)$
and $\mathrm{BL}_{Hyb}(A,C) \le \mathrm{BL}_{Hyb}(B,C)$.

3.4. Weighted hybrid binary logarithm similarity measures (WHBLSM) for SVNSs

Definition 7. Let $A = \langle x(T_A(x_i), I_P(x_i), F_P(x_i))\rangle$ and $B = \langle x(T_B(x_i), I_B(x_i), F_B(x_i))\rangle$ be any two SVNSs. The weighted hybrid binary logarithm similarity measure between SVNSs A and B is defined as follows:

$$\mathrm{BL}_{wHyb}(A,B) = \left[\begin{matrix}\lambda\left\{\sum_{i=1}^{n} w_i \log_2\left(2-\left(\frac{1}{3}\left(\begin{matrix}|T_A(x_i)-T_B(x_i)|\\ +|I_A(x_i)-I_B(x_i)|\\ +|F_A(x_i)-F_B(x_i)|\end{matrix}\right)\right)\right)\right\}\\ +(1-\lambda)\sum_{i=1}^{n} w_i \log_2\left(2-\max\left(\begin{matrix}|T_A(x_i)-T_B(x_i)|,\\ |I_A(x_i)-I_B(x_i)|,\\ |F_A(x_i)-F_B(x_i)|\end{matrix}\right)\right)\end{matrix}\right] \quad (6)$$

Here, $0 \le \lambda \le 1$.

Theorem 5. The weighted hybrid binary logarithm similarity measure $\mathrm{BL}_{wHyb}(A,B)$ between any two SVNSs A and B satisfy the following properties:

P1. $0 \le \mathrm{BL}_{wHyb}(A,B) \le 1$

P 2. $\mathrm{BL}_{wHyb}(A,B) = 1$, if and only if $A = B$

P 3. $\mathrm{BL}_{wHyb}(A,B) = \mathrm{BL}_{wHyb}(B,A)$

P4. If C is a SVNS in X and $A \subseteq B \subseteq C$, then $\mathrm{BL}_{wHyb}(A,C) \le \mathrm{BL}_{wHyb}(A,B)$ and $\mathrm{BL}_{wHyb}(A,C) \le \mathrm{BL}_{wHyb}(B,C)$.

Proof 1.

From the definition of SVNS, we write,
$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ and
$0 \le T_B(x) + I_B(x) + F_B(x) \le 3$

$$\Rightarrow 0 \le \max\left(\begin{matrix}|T_A(x_i)-T_B(x_i)|,|I_A(x_i)-I_B(x_i)|,\\ |F_A(x_i)-F_B(x_i)|\end{matrix}\right) \le 1$$

$$\Rightarrow \begin{matrix}0 \le |T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)|\\ +|F_A(x_i)-F_B(x_i)| \le 3\end{matrix};$$

$$\Rightarrow 0 \le \mathrm{BL}_{wHyb}(A,B) \le 1.$$

Proof 2.

For any two SVNSs A and B,

for $A = B$, we have,
$\Rightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$
$\Rightarrow |T_A(x) - T_B(x)| = 0, |I_A(x) - I_B(x)| = 0,$
$|F_A(x) - F_B(x)| = 0$
$\Rightarrow \mathrm{BL}_{wHyb}(A, B) = 1.$

Conversely,

for $\mathrm{BL}_{wHyb}(A, B) = 1$, we have,
$|T_A(x) - T_B(x)| = 0, |I_A(x) - I_B(x)| = 0,$
$|F_A(x) - F_B(x)| = 0$
$\Rightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$
$\Rightarrow A = B.$

Proof 3.

For any two SVNSs A and B, we have,
$|T_A(x) - T_B(x)| = |T_B(x) - T_A(x)|,$
$|I_A(x) - I_B(x)| = |I_B(x) - I_A(x)|,$
$|F_A(x) - F_B(x)| = |F_B(x) - F_A(x)|$
$\Rightarrow \mathrm{BL}_{wHyb}(A, B) = \mathrm{BL}_{wHyb}(B, A).$

Proof 4.

For $A \subseteq B \subseteq C$, we have,
$T_A(x) \le T_B(x) \le T_C(x), I_A(x) \ge I_B(x) \ge I_C(x),$
$F_A(x) \ge F_B(x) \ge F_C(x)$ for all $x \in X$.
$\Rightarrow |T_A(x) - T_B(x)| \le |T_A(x) - T_C(x)|,$
$|T_B(x) - T_C(x)| \le |T_A(x) - T_C(x)|;$
$|I_A(x) - I_B(x)| \le |I_A(x) - I_C(x)|,$
$|I_B(x) - I_C(x)| \le |I_A(x) - I_C(x)|;$
$|F_A(x) - F_B(x)| \le |F_A(x) - F_C(x)|,$
$|F_B(x) - F_C(x)| \le |F_A(x) - F_C(x)|.$
$\Rightarrow \mathrm{BL}_{wHyb}(A, C) \le \mathrm{BL}_{wHyb}(A, B)$ and
$\mathrm{BL}_{wHyb}(A, C) \le \mathrm{BL}_{wHyb}(B, C).$

4. A new entropy measure for SVNSs

Entropy strategy [52] is an important contribution for determining indeterminate information. Zhang *et al*. [53] introduced the fuzzy entropy. Vlachos and Sergiadis [54] proposed entropy function for intuitionistic fuzzy sets. Majumder and Samanta [55] developed some entropy measures for SVNSs. When attribute weights are completely unknown to decision makers, the entropy measure is used to calculate attribute weights. In this paper, we define an entropy measure for determining unknown attribute weights.

Definition 8. The entropy function of a SVNS $P = \left\langle T_{ij}^P(x), I_{ij}^P(x), F_{ij}^P(x) \right\rangle$ (i = 1, 2, ..., m; j = 1, 2, ..., n) is defined as follows:

$$E_j(P) = 1 - \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}^P(x) + F_{ij}^P(x)\right)\left(1 - 2I_{ij}^P(x)\right)^2 \quad (7)$$

$$w_j = \frac{1 - E_j(P)}{n - \sum_{j=1}^{n} E_j(P)} \quad (8)$$

Here, $\sum_{j=1}^{n} w_j = 1$

Theorem 6. The entropy function $E_j(P)$ satisfies the following properties:

P1. $E_j(P) = 0$, if $T_{ij} = 1, F_{ij} = I_{ij} = 0$.

P2. $E_j(P) = 1$, if $\left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle = \left\langle 0.5, 0.5, 0.5 \right\rangle$.

P3. $E_j(P) \ge E_j(Q)$, if $T_{ij}^P + F_{ij}^P \le T_{ij}^Q + F_{ij}^Q$; $I_{ij}^P \ge I_{ij}^Q$.

P4. $E_j(P) = E_j(P^c)$.

Proof 1.

$T_{ij} = 1, F_{ij} = I_{ij} = 0$

$$\Rightarrow E_j(P) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left[(1 + 0)\right] = 1 - \frac{n}{n} = 0$$

Proof 2.

$\left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle = \left\langle 0.5, 0.5, 0.5 \right\rangle$.

$$\Rightarrow E_j(P) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left[(0.5 + 0.5) \times 0\right] = 1 - 0 = 1$$

Proof 3.

$T_{ij}^P + F_{ij}^P \le T_{ij}^Q + F_{ij}^Q$, $I_{ij}^P \ge I_{ij}^Q$

$$\Rightarrow \sum_{i=1}^{m}\left(T_{ij}^P + F_{ij}^P\right)\left(1 - 2I_{ij}^P\right)^2 \le \sum_{i=1}^{m}\left(T_{ij}^Q + F_{ij}^Q\right)\left(1 - 2I_{ij}^Q\right)^2$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}^P + F_{ij}^P\right)\left(1 - 2I_{ij}^P\right)^2 \le \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}^Q + F_{ij}^Q\right)\left(1 - 2I_{ij}^Q\right)^2$$

$$\Rightarrow 1 - \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}^P + F_{ij}^P\right)\left(1 - 2I_{ij}^P\right)^2 \ge 1 - \frac{1}{n}\sum_{i=1}^{m}\left(T_{ij}^Q + F_{ij}^Q\right)\left(1 - 2I_{ij}^Q\right)^2$$

$\Rightarrow E_j(P) \ge E_j(Q).$

Proof 4.

Since $\left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle^c = \left\langle F_{ij}, 1 - I_{ij}, T_{ij} \right\rangle$, we have

$E_j(P) = E_j(P^c)$.

5. MAGDM strategy based on weighted hybrid binary logarithm similarity measure for SVNSs

Assume that (P_1, P_2, ..., P_m) be the alternatives, (C_1, C_2, ..., C_n) be the attributes of each alternative, and {D_1, D_2, ..., D_r} be the decision makers. Decision makers provide the rating of alternatives based on the predefined attribute. Each decision maker constructs a neutrosophic decision matrix associated with the alternatives based on each attribute shown in Equation (9). Using the following steps, we present the MAGDM strategy (see figure 1) based on weighted hybrid binary logarithm similarity measure (WHBLSM).

Step 1: Determine the relation between the alternatives and the attributes

At first, each decision maker prepares decision matrix. The relation between alternatives P_i (i = 1, 2, ..., m) and the attribute C_j (j = 1, 2, ..., n) corresponding to each decision maker is presented in the Equation (9).

$D_r[P \mid C] =$

$$\begin{array}{c} \\ P_1 \\ P_2 \\ \vdots \\ P_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \langle T_{11}^{D_r}, I_{11}^{D_r}, F_{11}^{D_r} \rangle & \langle T_{12}^{D_r}, I_{12}^{D_r}, F_{12}^{D_r} \rangle & \cdots & \langle T_{1n}^{D_r}, I_{1n}^{D_r}, F_{1n}^{D_r} \rangle \\ \langle T_{21}^{D_r}, I_{21}^{D_r}, F_{21}^{D_r} \rangle & \langle T_{22}^{D_r}, I_{22}^{D_r}, F_{22}^{D_r} \rangle & \cdots & \langle T_{2n}^{D_r}, I_{2n}^{D_r}, F_{2n}^{D_r} \rangle \\ \cdots & \cdots & \ddots & \cdots \\ \langle T_{m1}^{D_r}, I_{m1}^{D_r}, F_{m1}^{D_r} \rangle & \langle T_{m2}^{D_r}, I_{m2}^{D_r}, F_{m2}^{D_r} \rangle & \cdots & \langle T_{mn}^{D_r}, I_{mn}^{D_r}, F_{mn}^{D_r} \rangle \end{pmatrix} \end{array} \tag{9}$$

Here, $\langle T_{ij}^{D_r}, I_{ij}^{D_r}, F_{ij}^{D_r} \rangle$ (i = 1, 2, ..., m; j = 1, 2, ..., n) is the single valued neutrosophic rating value of the alternative P_i with respect to the attribute C_j corresponding to the decision maker D_r.

Step 2: Determine the core decision matrix

We form a new decision matrix, called core decision matrix to combine all the decision maker's opinions into a group opinion. Core decision matrix minimizes the biasness which is imposed by different decision makers and hence credibility to the final decision increases. The core decision matrix is presented in Equation (10).

$D[P \mid C] =$

$$\begin{array}{c} \\ P_1 \\ P_2 \\ \vdots \\ P_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \cdots & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} \\ \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \cdots & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} \\ \cdots & \cdots & \ddots & \cdots \\ \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} & \cdots & \frac{\overset{r}{\underset{t=1}{\oplus}} \langle T_{11}^{D_t}, I_{11}^{D_t}, F_{11}^{D_t} \rangle}{r} \end{pmatrix} \end{array} \tag{10}$$

Step 3: Determine the ideal solution

The evaluation of attributes can be categorized into benefit attribute and cost attribute. An ideal alternative can be determined by using a maximum operator for the benefit attributes and a minimum operator for the cost attributes for determining the best value of each attribute among all the alternatives. An ideal alternative [42] is presented as follows:

$P^* = \{C_1^*, C_2^*, \ldots, C_m^*\}$.

where the benefit attribute is

$$C_j^* = \left\langle \max_i T_{C_j}^{(P_i)}, \min_i I_{C_j}^{(P_i)}, \min_i F_{C_j}^{(P_i)} \right\rangle \tag{11}$$

and the cost attribute is

$$C_j^* = \left\langle \min_i T_{C_j}^{(P_i)}, \max_i I_{C_j}^{(P_i)}, \max_i F_{C_j}^{(P_i)} \right\rangle \tag{12}$$

Step 4: Determine the attribute weights

Using Equation (8), determine the weights of the attribute.

Step 5: Determine the WHBLSM values

Using Equation (6), calculate the weighted similarity measures for each alternative.

Step 6: Ranking the priority

All the alternatives are preference ranked based on the decreasing order of calculated measure values. The highest value reflects the best alternative.

Step 7: End.

6. An illustrative example

Suppose that a state government wants to construct an ecotourism park for the development of state tourism and especially for mental refreshment of children. After initial screening, three potential spots namely, spot-1 (P_1), spot-2 (P_2), and spot-3 (P_3) remain for further selection. A team

of three decision makers, namely, D_1, D_2, and D_3 has been constructed for selecting the most suitable spot with respect to the following attributes.

- Ecology (C_1),
- Costs (C_2),
- Technical facility (C_3),
- Transport (C_4),
- Risk factors (C_5)

The steps of decision-making strategy to select the best potential spot to construct an eco-tourism park based on the proposed strategy are stated below:

6.1. Steps of MAGDM strategy

We present MAGDM strategy based on the proposed WHBLSM using the following steps.

Step 1: Determine the relation between alternatives and attributes

The relation between alternatives P_1, P_2 and P_3 and the attribute set $\{C_1, C_2, C_3, C_4, C_5\}$ corresponding to the set of decision makers $\{D_1, D_2, D_3\}$ are presented in Equations (13), (14), and (15).

$$D_1[P \mid C] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} \langle 0.7, 0.4, 0.4\rangle & \langle 0.7, 0.4, 0.3\rangle & \langle 0.8, 0.1, 0.1\rangle & \langle 0.7, 0.2, 0,1\rangle & \langle 0.6, 0.5, 0.5\rangle \\ \langle 0.4, 0.3, 0.6\rangle & \langle 0.5, 0.2, 0.5\rangle & \langle 0.6, 0.2, 0.2\rangle & \langle 0.7, 0.3, 0.3\rangle & \langle 0.4, 0.3, 0.4\rangle \\ \langle 0.4, 0.2, 0.3\rangle & \langle 0.8, 0.1, 0.3\rangle & \langle 0.5, 0.4, 0.4\rangle & \langle 0.5, 0.2, 0.2\rangle & \langle 0.7, 0.3, 0.2\rangle \end{pmatrix} \end{array} \tag{13}$$

$$D_2[P \mid C] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} \langle 0.5, 0.2, 0.3\rangle & \langle 0.7, 0.4, 0.4\rangle & \langle 0.8, 0.2, 0.2\rangle & \langle 0.5, 0.2, 0,2\rangle & \langle 0.5, 0.5, 0.4\rangle \\ \langle 0.5, 0.4, 0.4\rangle & \langle 0.5, 0.2, 0.4\rangle & \langle 0.5, 0.3, 0.3\rangle & \langle 0.8, 0.3, 0.3\rangle & \langle 0.4, 0.1, 0.4\rangle \\ \langle 0.4, 0.2, 0.5\rangle & \langle 0.8, 0.2, 0.2\rangle & \langle 0.5, 0.3, 0.3\rangle & \langle 0.7, 0.2, 0.2\rangle & \langle 0.7, 0.4, 0.2\rangle \end{pmatrix} \end{array} \tag{14}$$

$$D_3[P \mid C] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} \langle 0.7, 0.4, 0.3\rangle & \langle 0.8, 0.2, 0.1\rangle & \langle 0.6, 0.3, 0.3\rangle & \langle 0.7, 0.2, 0,5\rangle & \langle 0.5, 0.6, 0.5\rangle \\ \langle 0.6, 0.2, 0.3\rangle & \langle 0.5, 0.1, 0.3\rangle & \langle 0.7, 0.4, 0.4\rangle & \langle 0.5, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.4\rangle \\ \langle 0.6, 0.2, 0.3\rangle & \langle 0.6, 0.4, 0.2\rangle & \langle 0.5, 0.3, 0.3\rangle & \langle 0.7, 0.4, 0.2\rangle & \langle 0.5, 0.6, 0.4\rangle \end{pmatrix} \end{array} \tag{15}$$

Step 2: Determine the core decision matrix

Using Equation (10), we construct the core decision matrix for all decision makers shown in Equation (16).

$$D[P \mid C] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{pmatrix} \langle 0.984, 0.324, 0.332\rangle & \langle 0.988, 0.324, 0.232\rangle & \langle 0.989, 0.184, 0.184\rangle & \langle 0.956, 0.203, 0.219\rangle & \langle 0.961, 0.452, 0.219\rangle \\ \langle 0.938, 0.292, 0.420\rangle & \langle 0.956, 0.162, 0.395\rangle & \langle 0.979, 0.292, 0.292\rangle & \langle 0.989, 0.304, 0.334\rangle & \langle 0.908, 0.232, 0.404\rangle \\ \langle 0.949, 0.203, 0.359\rangle & \langle 0.994, 0.203, 0.232\rangle & \langle 0.956, 0.334, 0.334\rangle & \langle 0.984, 0.255, 0.203\rangle & \langle 0.984, 0.420, 0.255\rangle \end{pmatrix} \end{array} \tag{16}$$

Step 3: Determine the ideal solution

Here, C_3 and C_4 denote benefit attributes and C_1, C_2 and C_5 denote cost attributes. Using Equations (11) and (12), we calculate the ideal solutions as follows:

$$P^* = \left\{ \begin{array}{l} \langle 0.938, 0.324, 0.420\rangle, \langle 0.956, 0.324, 0.395\rangle, \\ \langle 0.989, 0.184, 0.184\rangle, \langle 0.989, 0.203, 0.203\rangle, \\ \langle 0.908, 0.452, 0.404\rangle \end{array} \right\}.$$

Step 4: Determine the attribute weights

Using Equation (8), we calculate the attribute weights as follows:

$[w_1, w_2, w_3, w_4, w_5] =$

[0.1680, 0.3300, 0.2285, 0.2485, 0.0250]

Step 5: Determine the weighted hybrid binary logarithm similarity measures

Using Equation (6), we calculate similarity values for alternatives shown in Table 1.

Step 6: Ranking the alternatives

Ranking order of alternatives is prepared as the descending order of similarity values. Highest value indicates the best alternative. Ranking results are shown in Table 1 for different values of λ.

Step 7. End.

7. Sensitivity analysis

In this section, we discuss the variation of ranking results (see Table 1) for different values of λ. From the results shown in Tables 1, we observe that the proposed strategy provides the same ranking order for different values of λ.

8. Comparison analysis

In this section, we solve the problem with different existing strategies [33, 37, 38, 56]. Outcomes are furnished in the Table 2 and figure 2.

9. Contributions of the proposed strategy

- We propose two types of binary logarithm similarity measures and their hybrid similarity measure for SVNS environment. We have proved their basic properties.
- To calculate unknown weights structure of attributes in SVNS environment, we have proposed a new entropy function.
- We develop a decision making strategy based on the proposed weighted hybrid binary logarithm similarity measure (WHBLSM).
- We have solved a illustrative example to show the feasibility, applicability, and effectiveness of the proposed strategy.

10. Conclusion

Conclusions in the paper are concise as follows:

1. We have proposed hybrid binary logarithm similarity measure and weighted hybrid binary logarithm similarity measure for dealing indeterminacy in decision making situation.
2. We have defined a new entropy function to determine unknown attribute weights.
3. We have developed a new MAGDM strategy based on the proposed weighted hybrid binary logarithm similarity measure.
4. We have presented a numerical example to illustrate the proposed strategy.
5. We have conducted a sensitivity analysis
6. We have presented comparative analyses between the obtained results from the proposed strategies and different existing strategies in the literature. The proposed weighted hybrid binary logarithm similarity measure can be applied to solve MAGDM problems in clustering analysis, pattern recognition, personnel selection, etc.
7. Future research can be continued to investigate the proposed similarity measures in neutrosophic hybrid environment for tackling uncertainty, inconsistency and indeterminacy in decision making. The concept of the paper can be applied in practical decision-making, supply chain management, data mining, cluster analysis, teacher selection etc.

Table 1 Ranking order for different values of λ.

Similarity measures	(λ)	Measure values	Ranking order
$BL_{wHyb}(P^*,P_i)$	0.10	$BL_{wHyb}(P^*,P_1)=0.9426$; $BL_{wHyb}(P^*,P_2)=0.9233$; $BL_{wHyb}(P^*,P_3)=0.9101$	$P_1 \succ P_2 \succ P_3$
$BL_{wHyb}(P^*,P_i)$	0.25	$BL_{wHyb}(P^*,P_1)=0.9479$; $BL_{wHyb}(P^*,P_2)=0.9296$; $BL_{wHyb}(P^*,P_3)=0.9153$	$P_1 \succ P_2 \succ P_3$
$BL_{wHyb}(P^*,P_i)$	0.40	$BL_{wHyb}(P^*,P_1)=0.9532$; $BL_{wHyb}(P^*,P_2)=0.9357$; $BL_{wHyb}(P^*,P_3)=0.9207$	$P_1 \succ P_2 \succ P_3$
$BL_{wHyb}(P^*,P_i)$	0.55	$BL_{wHyb}(P^*,P_1)=0.9585$; $BL_{wHyb}(P^*,P_2)=0.9419$; $BL_{wHyb}(P^*,P_3)=0.9260$	$P_1 \succ P_2 \succ P_3$
$BL_{wHyb}(P^*,P_i)$	0.70	$BL_{wHyb}(P^*,P_1)=0.9638$; $BL_{wHyb}(P^*,P_2)=0.9482$; $BL_{wHyb}(P^*,P_3)=0.9313$	$P_1 \succ P_2 \succ P_3$
$BL_{wHyb}(P^*,P_i)$	0.90	$BL_{wHyb}(P^*,P_1)=0.9708$; $BL_{wHyb}(P^*,P_2)=0.9565$; $BL_{wHyb}(P^*,P_3)=0.9384$	$P_1 \succ P_2 \succ P_3$

Table 2 Ranking order for different existing strategies

Similarity measures	Measure values for P_1, P_2 and P_3	Ranking order
Mondal and Pramanik [37]	0.8901, 0.8679, 0.8093	$P_1 \succ P_2 \succ P_3$
Ye [33]	0.8409, 0.8189, 0.7766	$P_1 \succ P_2 \succ P_3$
Biswas et al. [56] (λ=0.55)	0.9511, 0.9219, 0.9007	$P_1 \succ P_2 \succ P_3$
Ye and Fu [38]	0.9161, 0.8758, 0.7900	$P_1 \succ P_2 \succ P_3$
Proposed strategy (λ=0.55)	0.9585, 0.9419, 0.9260	$P_1 \succ P_2 \succ P_3$

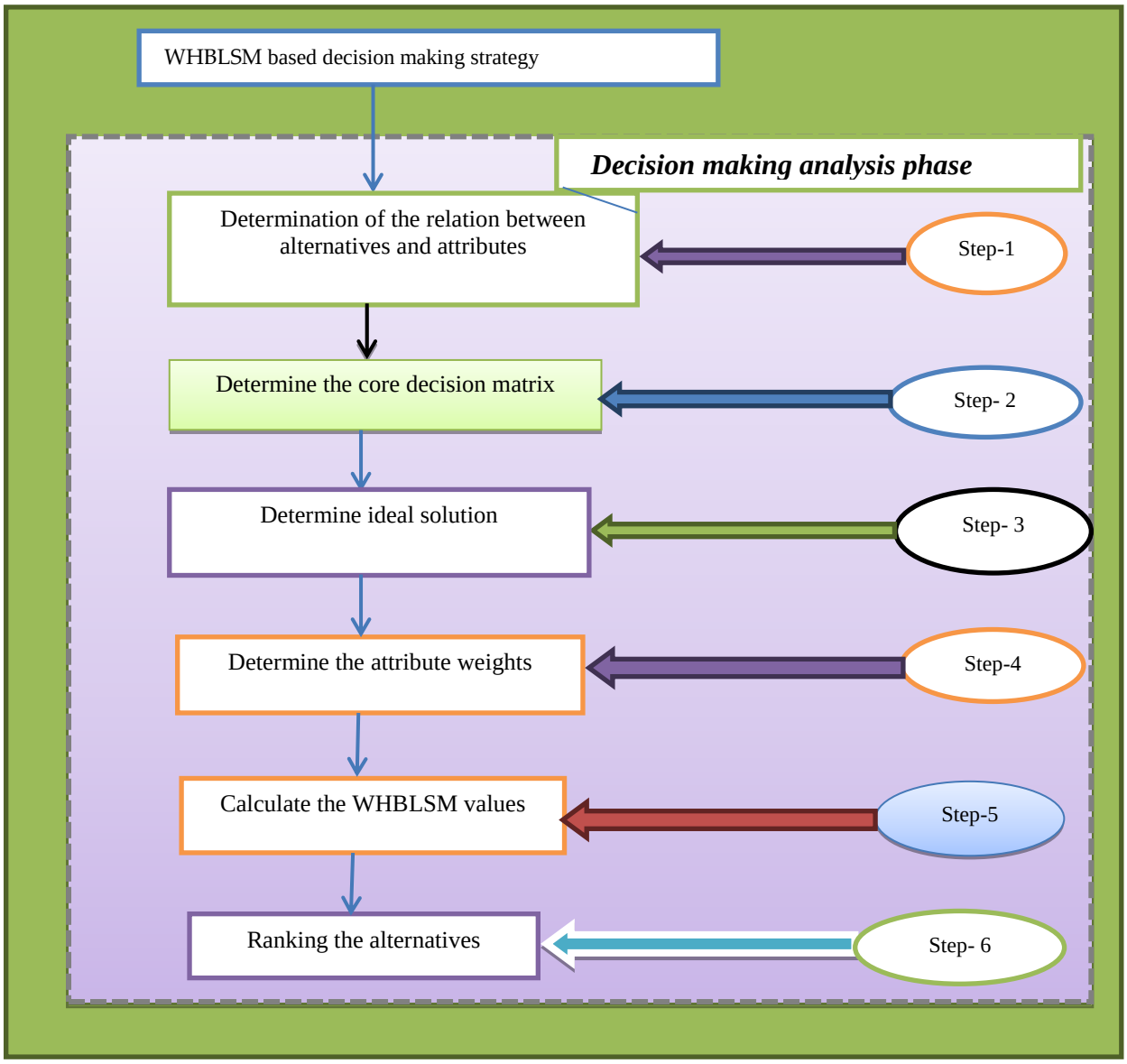

Fig. 1: Decision making phases of the proposed approach

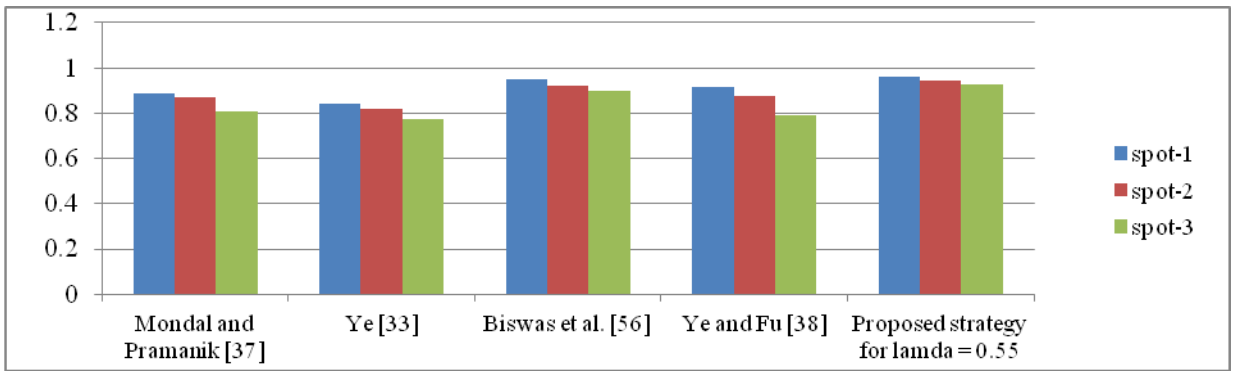

Fig. 2: Ranking order of different strategies

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Generalizations of Neutrosophic Subalgebras in BCK/BCI-Algebras Based on Neutrosophic Points

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Abstract: Saeid and Jun introduced the notion of neutrosophic points, and studied neutrosophic subalgebras of several types in BCK/BCI-algebras by using the notion of neutrosophic points (see [4] and [6]). More general form of neutrosophic points is considered in this paper, and generalizations of Saeid and Jun's results are discussed. The concepts of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are introduced, and several properties are investigated. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are discussed.

Keywords: $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

1 Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. As a more general platform which extends the notions of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued (intuitionistic) fuzzy set, Smarandache introduced the notion of neutrosophic sets (see [7, 8]), which is useful mathematical tool for dealing with incomplete, inconsistent and indeterminate information. For further particulars on neutrosophic set theory, we refer the readers to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

Jun [4] introduced the notion of (Φ, Ψ)-neutrosophic subalgebra of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigated related properties. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \vee q)$-neutrosophic subalgebra, and considered conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subalgebra. Saeid and Jun [6] gave relations between an $(\in, \in \vee q)$-neutrosophic subalgebra and a $(q, \in \vee q)$-neutrosophic subalgebra, and investigated properties on neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets.

The purpose of this article is to give an algebraic tool of neutrosophic set theory which can be used in applied sciences, for example, decision making problems, medical sciences etc. We consider a general form of neutrosophic points, and then we discuss generalizations of the papers [4] and [6]. As a generalization of $(\in, \in \vee q)$-neutrosophic subalgebras, we introduce the notions of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in BCK/BCI-algebras, and investigate several properties. We discuss characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We consider relations between $(\in, \in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

2 Preliminaries

By a *BCI-algebra*, we mean a set X with a binary operation $*$ and the special element 0 satisfying the conditions (see [3, 5]):

(a1) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)(x * x = 0)$,

(a4) $(\forall x, y \in X)(x * y = y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra* (see [3, 5]). A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X (see [3, 5]) if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [3] and [5] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [7]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\} \qquad (2.1)$$

where A_T, A_I and A_F are a truth membership function, an indeterminate membership function and a false membership function, respectively, from X into the unit interval $[0,1]$. The neutrosophic set (2.1) will be denoted by $A = (A_T, A_I, A_F)$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets (see [4]):

$$\begin{aligned}
&T_{\in}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_{\in}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_{\in}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are *neutrosophic $\in\vee q$-subsets*. It is clear that

$$T_{\in\vee q}(A;\alpha) = T_{\in}(A;\alpha) \cup T_q(A;\alpha), \tag{2.2}$$
$$I_{\in\vee q}(A;\beta) = I_{\in}(A;\beta) \cup I_q(A;\beta), \tag{2.3}$$
$$F_{\in\vee q}(A;\gamma) = F_{\in}(A;\gamma) \cup F_q(A;\gamma). \tag{2.4}$$

Given $\Phi, \Psi \in \{\in, q, \in\vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a (Φ, Ψ)-*neutrosophic subalgebra* of X (see [4]) if the following assertions are valid.

$$\begin{aligned}
&x \in T_{\Phi}(A;\alpha_x),\ y \in T_{\Phi}(A;\alpha_y)\\
&\qquad\Rightarrow\ x * y \in T_{\Psi}(A;\alpha_x \wedge \alpha_y),\\
&x \in I_{\Phi}(A;\beta_x),\ y \in I_{\Phi}(A;\beta_y)\\
&\qquad\Rightarrow\ x * y \in I_{\Psi}(A;\beta_x \wedge \beta_y),\\
&x \in F_{\Phi}(A;\gamma_x),\ y \in F_{\Phi}(A;\gamma_y)\\
&\qquad\Rightarrow\ x * y \in F_{\Psi}(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{2.5}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

3 Generalizations of $(\in, \in\vee q)$-neutrosophic subalgebras

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$\begin{aligned}
&T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\},\\
&I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\},\\
&F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\},\\
&T_{\in\vee q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\},\\
&I_{\in\vee q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\},\\
&F_{\in\vee q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}.
\end{aligned}$$

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic q_k-subsets*; and $T_{\in\vee q_{k_T}}(A;\alpha)$, $I_{\in\vee q_{k_I}}(A;\beta)$ and $F_{\in\vee q_{k_F}}(A;\gamma)$ are *neutrosophic $(\in\vee q_k)$-subsets*. For $\Phi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in\vee q, \in\vee q_k, \in\vee q_{k_T}, \in\vee q_{k_I}, \in\vee q_{k_F}\}$, the element of $T_{\Phi}(A;\alpha)$ (resp., $I_{\Phi}(A;\beta)$ and $F_{\Phi}(A;\gamma)$) is called a *neutrosophic T_{Φ}-point* (resp., *neutrosophic I_{Φ}-point* and *neutrosophic F_{Φ}-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee q_{k_T}}(A;\alpha) = T_{\in}(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$
$$I_{\in\vee q_{k_I}}(A;\beta) = I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$
$$F_{\in\vee q_{k_F}}(A;\gamma) = F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T^*_{\in}(A;\alpha) := \{x \in X \mid A_T(x) > \alpha\}, \tag{3.4}$$
$$I^*_{\in}(A;\beta) := \{x \in X \mid A_I(x) > \beta\}, \tag{3.5}$$
$$F^*_{\in}(A;\gamma) := \{x \in X \mid A_F(x) < \gamma\}. \tag{3.6}$$

Proposition 3.1. *For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we have*

$$\alpha \leq \tfrac{1-k}{2} \ \Rightarrow\ T_{q_k}(A;\alpha) \subseteq T^*_{\in}(A;\alpha), \tag{3.7}$$
$$\beta \leq \tfrac{1-k}{2} \ \Rightarrow\ I_{q_k}(A;\beta) \subseteq I^*_{\in}(A;\beta), \tag{3.8}$$
$$\gamma \geq \tfrac{1-k}{2} \ \Rightarrow\ F_{q_k}(A;\gamma) \subseteq F^*_{\in}(A;\gamma), \tag{3.9}$$
$$\alpha > \tfrac{1-k}{2} \ \Rightarrow\ T_{\in}(A;\alpha) \subseteq T_{q_k}(A;\alpha), \tag{3.10}$$
$$\beta > \tfrac{1-k}{2} \ \Rightarrow\ I_{\in}(A;\beta) \subseteq I_{q_k}(A;\beta), \tag{3.11}$$
$$\gamma < \tfrac{1-k}{2} \ \Rightarrow\ F_{\in}(A;\gamma) \subseteq F_{q_k}(A;\gamma). \tag{3.12}$$

Proof. If $\alpha \leq \frac{1-k}{2}$, then $1-\alpha \geq \frac{1+k}{2}$ and $\alpha \leq 1-\alpha$. Assume that $x \in T_{q_k}(A;\alpha)$. Then $A_T(x) + k > 1-\alpha \geq \frac{1+k}{2}$, and so $A_T(x) > \frac{1+k}{2} - k = \frac{1-k}{2} \geq \alpha$. Hence $x \in T^*_{\in}(A;\alpha)$. Similarly, we have the result (3.8). Suppose that $\gamma \geq \frac{1-k}{2}$ and let $x \in F_{q_k}(A;\gamma)$. Then $A_F(x) + \gamma + k < 1$, and thus

$$A_F(x) < 1 - \gamma - k \leq 1 - \tfrac{1-k}{2} - k = \tfrac{1-k}{2} \leq \gamma.$$

Hence $x \in F^*_{\in}(A;\gamma)$. Suppose that $\alpha > \frac{1-k}{2}$. If $x \in T_{\in}(A;\alpha)$,

then

$$A_T(x) + \alpha + k \geq 2\alpha + k > 2 \cdot \tfrac{1-k}{2} + k = 1$$

and so $x \in T_{q_k}(A;\alpha)$. Hence $T_{\in}(A;\alpha) \subseteq T_{q_k}(A;\alpha)$. Similarly, we can verify that if $\beta > \frac{1-k}{2}$, then $I_{\in}(A;\beta) \subseteq I_{q_k}(A;\beta)$. Suppose that $\gamma < \frac{1-k}{2}$. If $x \in F_{\in}(A;\gamma)$, then $A_F(x) \leq \gamma$, and thus

$$A_F(x) + \gamma + k \leq 2\gamma + k < 2 \cdot \tfrac{1-k}{2} + k = 1,$$

that is, $x \in F_{q_k}(A;\gamma)$. Hence $F_{\in}(A;\gamma) \subseteq F_{q_k}(A;\gamma)$. □

Definition 3.2. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \\ &\qquad \Rightarrow\ x * y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ &x \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \\ &\qquad \Rightarrow\ x * y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y), \\ &x \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \\ &\qquad \Rightarrow\ x * y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.13}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

An $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, \in \vee q_k)$-*neutrosophic subalgebra*.

Lemma 3.3 ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.14}$$

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_{q_{k_T}}(A;\alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. It follows from Lemma 3.3 that

$$\begin{aligned} &A_T(x*y) + \alpha + k_T \geq (A_T(x) \wedge A_T(y)) + \alpha + k_T \\ &= (A_T(x) + \alpha + k_T) \wedge (A_T(y) + \alpha + k_T) > 1 \end{aligned}$$

and so that $x*y \in T_{q_{k_T}}(A;\alpha)$. Hence $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of X. Similarly, we can prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of X. Now let $x, y \in F_{q_{k_F}}(A;\gamma)$. Then $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$, which imply from Lemma 3.3 that

$$\begin{aligned} &A_F(x*y) + \gamma + k_F \leq (A_F(x) \vee A_F(y)) + \gamma + k_F \\ &= (A_F(x) + \gamma + k_F) \vee (A_F(y) + \gamma + k_F) < 1. \end{aligned}$$

Hence $x * y \in F_{q_{k_F}}(A;\gamma)$ and so $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of X. □

Corollary 3.5. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_k}(A;\alpha)$, $I_{q_k}(A;\beta)$ and $F_{q_k}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

If we take $k_T = k_I = k_F = 0$ in Theorem 3.4, then we have the following corollary.

Corollary 3.6 ([4]). *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

Definition 3.7. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_{q_{k_T}}(A;\alpha_x),\ y \in T_{q_{k_T}}(A;\alpha_y) \\ &\qquad \Rightarrow\ x * y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ &x \in I_{q_{k_I}}(A;\beta_x),\ y \in I_{q_{k_I}}(A;\beta_y) \\ &\qquad \Rightarrow\ x * y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y), \\ &x \in F_{q_{k_F}}(A;\gamma_x),\ y \in F_{q_{k_F}}(A;\gamma_y) \\ &\qquad \Rightarrow\ x * y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

A $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called a $(q_k, \in \vee q_k)$-*neutrosophic subalgebra*.

Theorem 3.8. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ whenever they are nonempty.*

Proof. Let $x, y \in T_{q_{k_T}}(A;\alpha)$ for $\alpha \in (\frac{1-k_T}{2}, 1]$. Then $x * y \in T_{\in\vee q_{k_T}}(A;\alpha)$, that is, $x * y \in T_{\in}(A;\alpha)$ or $x * y \in T_{q_{k_T}}(A;\alpha)$. If $x * y \in T_{\in}(A;\alpha)$, then $x * y \in T_{q_{k_T}}(A;\alpha)$ by (3.10). Therefore $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of X. Similarly, we prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of X. Let $x, y \in F_{q_{k_F}}(A;\gamma)$ for $\gamma \in [0, \frac{1-k_F}{2})$. Then $x * y \in F_{\in\vee q_{k_F}}(A;\gamma)$, and so $x*y \in F_{\in}(A;\gamma)$ or $x*y \in F_{q_{k_F}}(A;\gamma)$. If $x*y \in F_{\in}(A;\gamma)$, then $x * y \in F_{q_{k_F}}(A;\gamma)$ by (3.12). Hence $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of X. □

Taking $k_T = k_I = k_F = 0$ in Theorem 3.8 induces the following corollary.

Corollary 3.9 ([4]). *If $A = (A_T, A_I, A_F)$ is a $(q, \in \vee q)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

We provide characterizations of an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

Theorem 3.10. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertion.*

$$\begin{aligned} A_T(x*y) &\geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\} \\ A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ A_F(x*y) &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \end{aligned} \tag{3.16}$$

for all $x, y \in X$.

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Assume that there exist $a, b \in X$ such that

$$A_T(a*b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}.$$

If $A_T(a) \wedge A_T(b) < \frac{1-k_T}{2}$, then $A_T(a*b) < A_T(a) \wedge A_T(b)$. Hence

$$A_T(a*b) < \alpha_t \leq A_T(a) \wedge A_T(b)$$

for some $\alpha_t \in (0, 1]$. It follows that $a \in T_\in(A; \alpha_t)$ and $b \in T_\in(A; \alpha_t)$ but $a*b \notin T_\in(A; \alpha_t)$. Moreover,

$$A_T(a*b) + \alpha_t < 2\alpha_t < 1 - k_T,$$

and so $a*b \notin T_{q_{k_T}}(A; \alpha_t)$. Thus $a*b \notin T_{\in\vee q_{k_T}}(A; \alpha_t)$, a contradiction. If $A_T(a) \wedge A_T(b) \geq \frac{1-k_T}{2}$, then $a \in T_\in(A; \frac{1-k_T}{2})$, $b \in T_\in(A; \frac{1-k_T}{2})$ and $a*b \notin T_\in(A; \frac{1-k_T}{2})$. Also,

$$A_T(a*b) + \tfrac{1-k_T}{2} < \tfrac{1-k_T}{2} + \tfrac{1-k_T}{2} = 1 - k_T,$$

i.e., $a*b \notin T_{q_{k_T}}(A; \frac{1-k_T}{2})$. Hence $a*b \notin T_{\in\vee q_{k_T}}(A; \frac{1-k_T}{2})$, a contradiction. Consequently,

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we know that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Suppose that there exist $a, b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [0, 1)$. If $A_F(a) \vee A_F(b) \geq \frac{1-k_F}{2}$, then

$$A_F(a*b) > \gamma_F \geq A_F(a) \vee A_F(b)$$

which implies that $a, b \in F_\in(A; \gamma_F)$ and $a*b \notin F_\in(A; \gamma_F)$. Also,

$$A_F(a*b) + \gamma_F > 2\gamma_F \geq 1 - k_F,$$

that is, $a*b \notin F_{q_{k_F}}(A; \gamma_F)$. Thus $a*b \notin F_{\in\vee q_{k_F}}(A; \gamma_F)$, which is a contradiction. If $A_F(a) \vee A_F(b) < \frac{1-k_F}{2}$, then $a, b \in F_\in(A; \frac{1-k_F}{2})$ and $a*b \notin F_\in(A; \frac{1-k_F}{2})$. Also,

$$A_F(a*b) + \tfrac{1-k_F}{2} > \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1 - k_F$$

and so $a*b \notin F_{q_{k_F}}(A; \frac{1-k_F}{2})$. Hence $a*b \notin F_{\in\vee q_{k_F}}(A; \frac{1-k_F}{2})$, a contradiction. Therefore

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.16). Let $x, y \in X$ and $\beta_x, \beta_y \in (0, 1]$ be such that $x \in I_\in(A; \beta_x)$ and $y \in I_\in(A; \beta_y)$. Then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \bigwedge\{\beta_x, \beta_y, \tfrac{1-k_I}{2}\}.$$

Suppose that $\beta_x \leq \frac{1-k_I}{2}$ or $\beta_y \leq \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \beta_x \wedge \beta_y$, and so $x*y \in I_\in(A; \beta_x \wedge \beta_y)$. Now, assume that $\beta_x > \frac{1-k_I}{2}$ and $\beta_y > \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \frac{1-k_I}{2}$, and so

$$A_I(x*y) + \beta_x \wedge \beta_y > \tfrac{1-k_I}{2} + \tfrac{1-k_I}{2} = 1 - k_I,$$

that is, $x*y \in I_{q_{k_I}}(A; \beta_x \wedge \beta_y)$. Hence

$$x*y \in I_{\in\vee q_{k_I}}(A; \beta_x \wedge \beta_y).$$

Similarly, we can verify that if $x \in T_\in(A; \alpha_x)$ and $y \in T_\in(A; \alpha_y)$, then $x*y \in T_{\in\vee q_{k_T}}(A; \alpha_x \wedge \alpha_y)$. Finally, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $x \in F_\in(A; \gamma_x)$ and $y \in F_\in(A; \gamma_y)$. Then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \bigvee\{\gamma_x, \gamma_y, \tfrac{1-k_F}{2}\}.$$

If $\gamma_x \geq \frac{1-k_F}{2}$ or $\gamma_y \geq \frac{1-k_F}{2}$, then $A_F(x*y) \leq \gamma_x \vee \gamma_y$ and thus $x*y \in F_\in(A; \gamma_x \vee \gamma_y)$. If $\gamma_x < \frac{1-k_F}{2}$ and $\gamma_y < \frac{1-k_F}{2}$, then $A_F(x*y) \leq \frac{1-k_F}{2}$. Hence

$$A_F(x*y) + \gamma_x \vee \gamma_y < \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1 - k_F,$$

that is, $x*y \in F_{q_{k_F}}(A; \gamma_x \vee \gamma_y)$. Thus

$$x*y \in F_{\in\vee q_{k_F}}(A; \gamma_x \vee \gamma_y).$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{k_F})$-neutrosophic subalgebra of X. □

Corollary 3.11 ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{pmatrix}.$$

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.10. □

Theorem 3.12. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $\beta \in (0, \frac{1-k_I}{2}]$ and $x, y \in I_\in(A;\beta)$. Then $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. It follows from Theorem 3.10 that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta \wedge \tfrac{1-k_I}{2} = \beta$$

and so that $x * y \in I_\in(A;\beta)$. Hence $I_\in(A;\beta)$ is a subalgebra of X for all $\beta \in (0, \frac{1-k_I}{2}]$. Similarly, we know that $T_\in(A;\alpha)$ is a subalgebra of X for all $\alpha \in (0, \frac{1-k_T}{2}]$. Let $\gamma \in [\frac{1-k_F}{2}, 1)$ and $x, y \in F_\in(A;\gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Using Theorem 3.10 implies that

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \gamma \vee \tfrac{1-k_F}{2} = \gamma.$$

Hence $x * y \in F_\in(A;\gamma)$, and therefore $F_\in(A;\gamma)$ is a subalgebra of X for all $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, suppose that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $a, b \in X$ such that

$$A_T(a * b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\},$$

then $a, b \in T_\in(A;\alpha_T)$ by taking

$$\alpha_T := \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}.$$

Since $T_\in(A;\alpha_T)$ is a subalgebra of X, it follows that $a * b \in T_\in(A;\alpha_T)$, that is, $A_T(a * b) \geq \alpha_T$. This is a contradiction, and hence

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now, assume that there exist $a, b \in X$ such that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a * b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [\frac{1-k_F}{2}, 1)$. Hence $a, b \in F_\in(A;\gamma_F)$, and so $a * b \in F_\in(A;\gamma_F)$ since $F_\in(A;\gamma_F)$ is a subalgebra of X. It follows that $A_F(a * b) \leq \gamma_F$ which is a contradiction. Thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.10. □

Corollary 3.13. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ whenever they are nonempty.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.12. □

Theorem 3.14. *Every $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.*

Proof. Straightforward. □

The converse of Theorem 3.14 is not true as seen in the following example.

Example 3.15. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1 (see [5]).

Table 1: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}_I(X)$ defined by Table 2

Table 2: Tabular representation of "$A = (A_T, A_I, A_F)$"

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.2
a	0.7	0.3	0.6
b	0.3	0.6	0.6
c	0.3	0.3	0.4

If $k_T = 0.36$, then

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.32]. \end{cases}$$

If $k_I = 0.32$, then

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.34]. \end{cases}$$

If $k_F = 0.36$, then

$$F_{\in}(A;\gamma) = \begin{cases} \{0\} & \text{if } \gamma \in [0.32, 0.4), \\ \{0, c\} & \text{if } \gamma \in [0.4, 0.6), \\ X & \text{if } \gamma \in [0.6, 1]. \end{cases}$$

We know that $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, 0.32]$, $\beta \in (0, 0.34]$ and $\gamma \in [0.32, 1)$. It follows from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.36$, $k_I = 0.32$ and $k_F = 0.36$. Since

$$A_T(0) = 0.6 < 0.7 = A_T(a) \wedge A_T(a)$$

and/or

$$A_I(0) = 0.5 > 0.3 = A_I(c) \vee A_I(c),$$

we know that $A = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of X by Lemma 3.3.

Definition 3.16. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \\ &\qquad \Rightarrow\ x * y \in T_{q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ &x \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \\ &\qquad \Rightarrow\ x * y \in I_{q_{k_I}}(A;\beta_x \wedge \beta_y), \\ &x \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \\ &\qquad \Rightarrow\ x * y \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.17}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

An $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, q_k)$-*neutrosophic subalgebra*.

Theorem 3.17. *Every $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.*

Proof. Straightforward. □

The converse of Theorem 3.17 is not true as seen in the following example.

Example 3.18. Consider the BCI-algebra $X = \{0, a, b, c\}$ and the neutrosophic set $A = (A_T, A_I, A_F)$ which are given in Example 3.15. Taking $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$ imply that

$$T_{\in}(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.4], \end{cases}$$

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.35], \end{cases}$$

and

$$F_{\in}(A;\gamma) = \begin{cases} \{0\} & \text{if } \beta \in [0.38, 0.4), \\ \{0, c\} & \text{if } \beta \in [0.4, 0.6), \\ X & \text{if } \beta \in [0.6, 1). \end{cases}$$

Since X, $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ are subalgebras of X, we know from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$. Note that

$$a * b \notin T_{q_{0.2}}(A; 0.25 \wedge 0.4)$$

for $a \in T_{\in}(A; 0.4)$ and $b \in T_{\in}(A; 0.25)$,

$$b * c \notin I_{q_{0.3}}(A; 0.5 \wedge 0.27)$$

for $b \in I_{\in}(A; 0.5)$ and $c \in I_{\in}(A; 0.27)$, and/or

$$a * c \notin F_{q_{0.24}}(A; 0.6 \vee 0.44)$$

for $a \in F_{\in}(A; 0.6)$ and $c \in F_{\in}(A; 0.44)$. Hence $A = (A_T, A_I, A_F)$ is not an $(\in, q_{(0.2,0.3,0.24)})$-neutrosophic subalgebra of X.

Theorem 3.19. *If $0 \le k_T < j_T < 1$, $0 \le k_I < j_I < 1$ and $0 \le j_F < k_F < 1$, then every $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra is an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra.*

Proof. Straightforward. □

The following example shows that if $0 \le k_T < j_T < 1$, $0 \le k_I < j_I < 1$ and $0 \le j_F < k_F < 1$, then an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra may not be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

Example 3.20. Let X be the BCI-algebra given in Example 3.15 and let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 3

Table 3: Tabular representation of "$A = (A_T, A_I, A_F)$"

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.42	0.40	0.44
a	0.40	0.44	0.66
b	0.48	0.36	0.66
c	0.40	0.36	0.33

If $k_T = 0.04$, then

$$T_{\in}(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42], \\ \{b\} & \text{if } \alpha \in (0.42, 0.48]. \end{cases}$$

Note that $T_{\in}(A;\alpha)$ is not a subalgebra of X for $\alpha \in (0.42, 0.48]$.

If $k_I = 0.08$, then

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40], \\ \{a\} & \text{if } \beta \in (0.40, 0.44], \\ \emptyset & \text{if } \beta \in (0.44, 0.46]. \end{cases}$$

Note that $I_{\in}(A;\beta)$ is not a subalgebra of X for $\beta \in (0.40, 0.44]$.

If $k_F = 0.42$, then

$$F_{\in}(A;\gamma) = \begin{cases} \emptyset & \text{if } \gamma \in [0.29, 0.33), \\ \{c\} & \text{if } \gamma \in [0.33, 0.44), \\ \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Note that $F_{\in}(A;\gamma)$ is not a subalgebra of X for $\gamma \in [0.33, 0.44)$. Therefore $A = (A_T, A_I, A_F)$ is not an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.04$, $k_I = 0.08$ and $k_F = 0.42$.

If $j_T = 0.16$, then

$$T_{\in}(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42]. \end{cases}$$

If $j_I = 0.20$, then

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40]. \end{cases}$$

If $j_F = 0.12$, then

$$F_{\in}(A;\gamma) = \begin{cases} \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra of X for $j_T = 0.16$, $j_I = 0.20$ and $j_F = 0.12$.

Given a subset S of X, consider a neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ in X defined by

$$A_S(x) := \begin{cases} (1, 1, 0) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_{ST}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{SI}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{SF}(x) := \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3.21. *A nonempty subset S of $X \in \mathcal{B}(X)$ is a subalgebra of X if and only if the neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

Proof. Let S be a subalgebra of X. Then neutrosophic $\in$-subsets $T_{\in}(A_{ST};\alpha)$, $I_{\in}(A_{ST};\beta)$ and $F_{\in}(A_{ST};\gamma)$ are obviously subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Hence $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.12.

Conversely, assume that $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $x, y \in S$. Then

$$\begin{aligned} A_{ST}(x * y) &\geq \bigwedge\{A_{ST}(x), A_{ST}(y), \tfrac{1-k_T}{2}\} \\ &= 1 \wedge \tfrac{1-k_T}{2} = \tfrac{1-k_T}{2}, \end{aligned}$$

$$\begin{aligned} A_{SI}(x * y) &\geq \bigwedge\{A_{SI}(x), A_{SI}(y), \tfrac{1-k_I}{2}\} \\ &= 1 \wedge \tfrac{1-k_I}{2} = \tfrac{1-k_I}{2} \end{aligned}$$

and

$$\begin{aligned} A_{SF}(x * y) &\leq \bigvee\{A_{SF}(x), A_{SF}(y), \tfrac{1-k_F}{2}\} \\ &= 0 \vee \tfrac{1-k_F}{2} = \tfrac{1-k_F}{2}, \end{aligned}$$

which imply that

$$A_{ST}(x * y) = 1,\ A_{SI}(x * y) = 1 \text{ and } A_{SF}(x * y) = 0.$$

Hence $x * y \in S$, and so S is a subalgebra of X. □

Theorem 3.22. *Let S be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, there exists an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of X such that $T_{\in}(A;\alpha) = S$, $I_{\in}(A;\beta) = S$ and $F_{\in}(A;\gamma) = S$.*

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by

$$A(x) := \begin{cases} (\alpha, \beta, \gamma) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_T(x) := \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_I(x) := \begin{cases} \beta & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_F(x) := \begin{cases} \gamma & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $T_{\in}(A;\alpha) = S$, $I_{\in}(A;\beta) = S$ and $F_{\in}(A;\gamma) = S$. Suppose that

$$A_T(a*b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}$$

for some $a, b \in X$. Since $\#\mathrm{Im}(A_T) = 2$, it follows that $\bigwedge\{A_T(a), A_T(b), \frac{1-k_T}{2}\} = \alpha$ and $A_T(a*b) = 0$. Hence $A_T(a) = \alpha = A_T(b)$, and so $a, b \in S$. Since S is a subalgebra of X, we have $a*b \in S$. Thus $A_T(a*b) = \alpha$, a contradiction. Therefore

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Assume that there exist $a, b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) = 1$ and $\bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\} = \gamma$ since $\#\mathrm{Im}(A_F) = 2$. It follows that $A_F(a) = \gamma = A_F(b)$ and so that $a, b \in S$. Hence $a*b \in S$, and so $A_F(a*b) = \gamma$, which is a contradiction. Thus

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.10. □

Corollary 3.23. *Let S be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, 0.5]$, $\beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, there exists an $(\in, \in \vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of X such that $T_{\in}(A;\alpha) = S$, $I_{\in}(A;\beta) = S$ and $F_{\in}(A;\gamma) = S$.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.22. □

Theorem 3.24. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

(2) *The neutrosophic $(\in \ \vee q_k)$-subsets $T_{\in \vee q_{k_T}}(A;\alpha)$, $I_{\in \vee q_{k_I}}(A;\beta)$ and $F_{\in \vee q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $x, y \in I_{\in \vee q_{k_I}}(A;\beta)$ for $\beta \in (0, 1]$. Then $A_I(x) \geq \beta$ or $A_I(x) + \beta + k_I > 1$, and $A_I(y) \geq \beta$ or $A_I(y) + \beta + k_I > 1$. Using Theorem 3.10, we have

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}.$$

Case 1. $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. If $\beta > \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} = \tfrac{1-k_I}{2},$$

and so $A_I(x*y) + \beta > \frac{1-k_I}{2} + \frac{1-k_I}{2} = 1 - k_I$. Hence $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta,$$

and thus $x*y \in I_{\in}(A;\beta)$. Hence

$$x*y \in I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta) = I_{\in \vee q_{k_I}}(A;\beta).$$

Case 2. $A_I(x) \geq \beta$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &= A_I(y) \wedge \tfrac{1-k_I}{2} > (1 - \beta - k_I) \wedge \tfrac{1-k_I}{2} \\ &= 1 - \beta - k_I, \end{aligned}$$

and so $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &\geq \bigwedge\{\beta, 1 - \beta - k_I, \tfrac{1-k_I}{2}\} = \beta, \end{aligned}$$

and thus $x*y \in I_{\in}(A;\beta)$. Therefore $x*y \in I_{\in \vee q_{k_I}}(A;\beta)$.

Case 3. $A_I(x) + \beta + k_I > 1$ and $A_I(y) \geq \beta$. We have $x*y \in I_{\in \vee q_{k_I}}(A;\beta)$ by the similar way to the Case 2.

Case 4. $A_I(x) + \beta + k_I > 1$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then $1 - \beta - k_I < \frac{1-k_I}{2}$, and so

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} > 1 - \beta - k_I,$$

i.e., $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &\geq (1 - \beta - k_I) \wedge \tfrac{1-k_I}{2} \\ &= \tfrac{1-k_I}{2} \geq \beta, \end{aligned}$$

i.e., $x*y \in I_{\in}(A;\beta)$. Hence $x*y \in I_{\in \vee q_{k_I}}(A;\beta)$. Consequently, $I_{\in \vee q_{k_I}}(A;\beta)$ is a subalgebra of X. Similarly, we can prove that if $x, y \in T_{\in \vee q_{k_T}}(A;\alpha)$ for $\alpha \in (0, 1]$, then $x*y \in T_{\in \vee q_{k_T}}(A;\alpha)$, that is, $T_{\in \vee q_{k_T}}(A;\alpha)$ is a subalgebra of X. Let $x, y \in F_{\in \vee q_{k_F}}(A;\gamma)$ for $\gamma \in [0, 1)$. Then $A_F(x) \leq \gamma$ or $A_F(x) + \gamma + k_F < 1$, and $A_F(y) \leq \gamma$ or $A_F(y) + \gamma + k_F < 1$. Using Theorem 3.10, we have

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}.$$

Case 1. $A_F(x) \le \gamma$ and $A_F(y) \le \gamma$. If $\gamma < \frac{1-k_F}{2}$, then

$$A_F(x*y) \le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} = \tfrac{1-k_F}{2},$$

and so $A_F(x*y) + \gamma < \frac{1-k_F}{2} + \frac{1-k_F}{2} = 1 - k_F$. Hence $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \ge \frac{1-k_F}{2}$, then

$$A_F(x*y) \le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \le \gamma,$$

and thus $x*y \in F_{\in}(A;\gamma)$. Hence

$$x*y \in F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma) = F_{\in\vee q_{k_F}}(A;\gamma).$$

Case 2. $A_F(x) \le \gamma$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &= A_F(y) \vee \tfrac{1-k_F}{2} < (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} \\ &= 1-\gamma-k_F, \end{aligned}$$

and so $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \ge \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &\le \bigvee\{\gamma, 1-\gamma-k_F, \tfrac{1-k_F}{2}\} = \gamma, \end{aligned}$$

and thus $x*y \in F_{\in}(A;\gamma)$. Therefore $x*y \in F_{\in\vee q_{k_F}}(A;\gamma)$.

Similarly, if $A_I(x) + \beta + k_I < 1$ and $A_I(y) \le \beta$, then $x*y \in F_{\in\vee q_{k_F}}(A;\gamma)$.

Finally, assume that $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then $1-\gamma-k_F > \frac{1-k_F}{2}$, and so

$$A_F(x*y) \le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} < 1-\gamma-k_F,$$

i.e., $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \ge \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\le \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &\le (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} \\ &= \tfrac{1-k_F}{2} \le \gamma, \end{aligned}$$

i.e., $x*y \in F_{\in}(A;\gamma)$. Hence $x*y \in F_{\in\vee q_{k_F}}(A;\gamma)$. Therefore $F_{\in\vee q_{k_F}}(A;\gamma)$ is a subalgebra of X.

Conversely, suppose that (2) is valid. If it is possible, let

$$A_T(x*y) < \alpha \le \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for some $\alpha \in (0, \frac{1-k_T}{2})$. Then

$$x, y \in T_{\in}(A;\alpha) \subseteq T_{\in\vee q_{k_T}}(A;\alpha),$$

which implies that $x*y \in T_{\in\vee q_{k_T}}(A;\alpha)$. Thus $A_T(x*y) \ge \alpha$ or $A_T(x*y) + \alpha + k_T > 1$, a contradiction. Hence

$$A_T(x*y) \ge \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x*y) \ge \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now assume that there exist $a, b \in X$ and $\gamma \in (\frac{1-k_F}{2}, 1)$ such that

$$A_F(a*b) > \gamma \ge \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $a, b \in F_{\in}(A;\gamma) \subseteq F_{\in\vee q_{k_F}}(A;\gamma)$, which implies that

$$a*b \in F_{\in\vee q_{k_F}}(A;\gamma).$$

Thus $A_F(a*b) \le \gamma$ or $A_F(a*b) + \gamma + k_F < 1$, which is a contradiction. Hence

$$A_F(x*y) \ge \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Using Theorem 3.10, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. □

4 Conclusions

Neutrosophic set theory is a nice mathematical tool which can be applied to several fields. The aim of this paper is to consider a general form of neutrosophic points, and to discuss generalizations of the papers [4] and [6]. We have introduce the notions of $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in BCK/BCI-algebras, and have investigated several properties. We have discussed characterizations of $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We have considered relations between $(\in,\in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We hope the idea and result in this paper can be a mathematical tool for dealing with several informations containing uncertainty such as medical diagnosis, decision making, graph theory, etc. So, based on the results in this article, our future research will be focused to solve real-life problems under the opinions of experts in a neutrosophic set environment such as medical diagnosis, decision making, graph theory etc. In particular, Bucolo et al. [2] suggested a generalization of the synchronization principles for the class of array of fuzzy logic chaotic based dynamical systems and evaluated as alternative approach to build locally connected fuzzy complex systems by manipulating both the rules driving the cells and the architecture of the system. We will also try to study complex dynamics through neutrosophic environment. The future works also may use the study neutrosophic set environment on several related algebraic structures, for example, MV-algebras, BL-algebras, R_0-algebras, EQ-algebras, equality algebras, MTL-algebras etc.

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Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras

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Abstract: Characterizations of an $(\in, \in)$-neutrosophic ideal are considered. Any ideal in a BCK/BCI-algebra will be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. The relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra is discussed. Conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal are provided. Using a collection of ideals in a BCK/BCI-algebra, an $(\in, \in)$-neutrosophic ideal is established. Equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals are introduced, and related properties are investigated.

Keywords: $(\in, \in)$-neutrosophic subalgebra, $(\in, \in)$-neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [8, 9, 10] introduced neutrosophic set (NS) as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site

http://fs.gallup.unm.edu/neutrosophy.htm.

Jun et al. studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [5] and [7]).

In this paper, we characterize an $(\in, \in)$-neutrosophic ideal in a BCK/BCI-algebra. We show that any ideal in a BCK/BCI-algebra can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. We investigate the relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra. We provide conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal. Using a collection of ideals in a BCK/BCI-algebra, we establish an $(\in, \in)$-neutrosophic ideal. We discuss equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals, and investigate related properties.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\ (x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\begin{pmatrix} x \le y \ \Rightarrow\ x * z \le y * z \\ x \le y \ \Rightarrow\ z * y \le z * x \end{pmatrix}, \tag{2.2}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.6}$$

We refer the reader to the books [4, 6] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [9]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned} T_\in(A_\sim; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\}, \\ I_\in(A_\sim; \beta) &:= \{x \in X \mid A_I(x) \geq \beta\}, \\ F_\in(A_\sim; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are *neutrosophic $\in$-subsets*.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic subalgebra* of X (see [5]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x * y \in T_\in(A_\sim; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x * y \in I_\in(A_\sim; \beta_x \wedge \beta_y), \\ x \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x * y \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic ideal* of X (see [7]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x) \ \Rightarrow\ 0 \in T_\in(A_\sim; \alpha_x) \\ x \in I_\in(A_\sim; \beta_x) \ \Rightarrow\ 0 \in I_\in(A_\sim; \beta_x) \\ x \in F_\in(A_\sim; \gamma_x) \ \Rightarrow\ 0 \in F_\in(A_\sim; \gamma_x) \end{array}\right) \tag{2.8}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.9}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

3 $(\in, \in)$-neutrosophic subalgebras and ideals

We first provide characterizations of an $(\in, \in)$-neutrosophic ideal.

Theorem 3.1. *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(0) \geq A_T(x), \\ A_I(0) \geq A_I(x), \\ A_F(0) \leq A_F(x) \end{array}\right) \tag{3.1}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.2}$$

Proof. Assume that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Suppose there exist $a, b, c \in X$ be such that $A_T(0) < A_T(a)$, $A_I(0) < A_I(b)$ and $A_F(0) > A_F(c)$. Then $a \in T_\in(A_\sim; A_T(a))$, $b \in I_\in(A_\sim; A_I(b))$ and $c \in F_\in(A_\sim; A_F(c))$. But

$$0 \notin T_\in(A_\sim; A_T(a)) \cap I_\in(A_\sim; A_I(b)) \cap F_\in(A_\sim; A_F(c)).$$

This is a contradiction, and thus $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that $A_T(x) < A_T(x * y) \wedge A_T(y)$, $A_I(a) < A_I(a * b) \wedge A_I(b)$ and $A_F(c) > A_F(c * d) \vee A_F(d)$ for some $x, y, a, b, c, d \in X$. Taking $\alpha := A_T(x*y) \wedge A_T(y)$, $\beta := A_I(a*b) \wedge A_I(b)$ and $\gamma := A_F(c*d) \vee A_F(d)$ imply that $x*y \in T_\in(A_\sim; \alpha)$, $y \in T_\in(A_\sim; \alpha)$, $a * b \in I_\in(A_\sim; \beta)$, $b \in I_\in(A_\sim; \beta)$, $c * d \in F_\in(A_\sim; \gamma)$ and $d \in F_\in(A_\sim; \gamma)$. But $x \notin T_\in(A_\sim; \alpha)$, $a \notin I_\in(A_\sim; \beta)$ and $c \notin F_\in(A_\sim; \gamma)$. This is impossible, and so (3.2) is valid.

Conversely, suppose $A_\sim = (A_T, A_I, A_F)$ satisfies two conditions (3.1) and (3.2). For any $x, y, z \in X$, let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A_\sim; \alpha)$, $y \in I_\in(A_\sim; \beta)$ and

$z \in F_{\in}(A_\sim;\gamma)$. It follows from (3.1) that $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(y) \geq \beta$ and $A_F(0) \leq A_F(z) \leq \gamma$ and so that $0 \in T_{\in}(A_\sim;\alpha) \cap I_{\in}(A_\sim;\beta) \cap F_{\in}(A_\sim;\gamma)$. Let $a, b, c, d, x, y \in X$ be such that $a * b \in T_{\in}(A_\sim;\alpha_a)$, $b \in T_{\in}(A_\sim;\alpha_b)$, $c * d \in I_{\in}(A_\sim;\beta_c)$, $d \in I_{\in}(A_\sim;\beta_d)$, $x * y \in F_{\in}(A_\sim;\gamma_x)$, and $y \in F_{\in}(A_\sim;\gamma_y)$ for $\alpha_a, \alpha_b, \beta_c, \beta_d \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$. Using (3.2), we have

$$\begin{aligned} A_T(a) &\geq A_T(a*b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b \\ A_I(c) &\geq A_I(c*d) \wedge A_I(d) \geq \beta_c \wedge \beta_d \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) \leq \gamma_x \vee \gamma_y. \end{aligned}$$

Hence $a \in T_{\in}(A_\sim;\alpha_a \wedge \alpha_b)$, $c \in I_{\in}(A_\sim;\beta_c \wedge \beta_d)$ and $x \in F_{\in}(A_\sim;\gamma_x \vee \gamma_y)$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

Theorem 3.2. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X. Then the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *The nonempty neutrosophic $\in$-subsets $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X and assume that $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are nonempty for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x, y, z \in X$ such that $x \in T_{\in}(A_\sim;\alpha)$, $y \in I_{\in}(A_\sim;\beta)$ and $z \in F_{\in}(A_\sim;\gamma)$. It follows from (2.8) that

$$0 \in T_{\in}(A_\sim;\alpha) \cap I_{\in}(A_\sim;\beta) \cap F_{\in}(A_\sim;\gamma).$$

Let $x, y, a, b, u, v \in X$ be such that $x * y \in T_{\in}(A_\sim;\alpha)$, $y \in T_{\in}(A_\sim;\alpha)$, $a * b \in I_{\in}(A_\sim;\beta)$, $b \in I_{\in}(A_\sim;\beta)$, $u * v \in F_{\in}(A_\sim;\gamma)$ and $v \in F_{\in}(A_\sim;\gamma)$. Then

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) \geq \alpha \wedge \alpha = \alpha \\ A_I(a) &\geq A_I(a*b) \wedge A_I(b) \geq \beta \wedge \beta = \beta \\ A_F(u) &\leq A_F(u*v) \vee A_F(v) \leq \gamma \vee \gamma = \gamma \end{aligned}$$

by (3.2), and so $x \in T_{\in}(A_\sim;\alpha)$, $a \in I_{\in}(A_\sim;\beta)$ and $u \in F_{\in}(A_\sim;\gamma)$. Hence the nonempty neutrosophic $\in$-subsets $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are nonempty and are ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Assume that $A_T(0) < A_T(x)$, $A_I(0) < A_I(y)$ and $A_F(0) > A_F(z)$ for some $x, y, z \in X$. Then $x \in T_{\in}(A_\sim;A_T(x))$, $y \in I_{\in}(A_\sim;A_I(y))$ and $z \in F_{\in}(A_\sim;A_F(z))$, that is, $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are nonempty. But $0 \notin T_{\in}(A_\sim;A_T(x)) \cap I_{\in}(A_\sim;A_I(y)) \cap F_{\in}(A_\sim;A_F(z))$, which is a contradiction since $T_{\in}(A_\sim;A_T(x))$, $I_{\in}(A_\sim;A_I(y))$ and $F_{\in}(A_\sim;A_F(z))$ are ideals of X. Hence $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that

$$\begin{aligned} A_T(x) &< A_T(x*y) \wedge A_T(y), \\ A_I(a) &< A_I(a*b) \wedge A_I(b), \\ A_F(u) &> A_F(u*v) \vee A_F(v) \end{aligned}$$

for some $x, y, a, b, u, v \in X$. Taking $\alpha := A_T(x*y) \wedge A_T(y)$, $\beta := A_I(a*b) \wedge A_I(b)$ and $\gamma := A_F(u*v) \vee A_F(v)$ imply that $\alpha, \beta \in (0,1]$, $\gamma \in [0,1)$, $x * y \in T_{\in}(A_\sim;\alpha)$, $y \in T_{\in}(A_\sim;\alpha)$, $a * b \in I_{\in}(A_\sim;\beta)$, $b \in I_{\in}(A_\sim;\beta)$, $u * v \in F_{\in}(A_\sim;\gamma)$ and $v \in F_{\in}(A_\sim;\gamma)$. But $x \notin T_{\in}(A_\sim;\alpha)$, $a \notin I_{\in}(A_\sim;\beta)$ and $u \notin F_{\in}(A_\sim;\gamma)$. This is a contradiction since $T_{\in}(A_\sim;\alpha)$, $I_{\in}(A_\sim;\beta)$ and $F_{\in}(A_\sim;\gamma)$ are ideals of X. Thus

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Proposition 3.3. *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertions.*

$$(\forall x, y \in X)\left(x \leq y \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases}\right), \tag{3.3}$$

$$(\forall x, y, z \in X)\left(x * y \leq z \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \wedge A_T(z) \\ A_I(x) \geq A_I(y) \wedge A_I(z) \\ A_F(x) \leq A_F(y) \vee A_F(z) \end{cases}\right). \tag{3.4}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y) \end{aligned}$$

by Theorem 3.1. Hence (3.3) is valid. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x*y)*z = 0$, and thus

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) \\ &\geq (A_T((x*y)*z) \wedge A_T(z)) \wedge A_T(y) \\ &\geq (A_T(0) \wedge A_T(z)) \wedge A_T(y) \\ &\geq A_T(z) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &\geq A_I(x*y) \wedge A_I(y) \\ &\geq (A_I((x*y)*z) \wedge A_I(z)) \wedge A_I(y) \\ &\geq (A_I(0) \wedge A_I(z)) \wedge A_I(y) \\ &\geq A_I(z) \wedge A_I(y) \end{aligned}$$

and

$$\begin{aligned} A_F(x) &\leq A_F(x*y) \vee A_F(y) \\ &\leq (A_F((x*y)*z) \vee A_F(z)) \vee A_F(y) \\ &\leq (A_F(0) \vee A_F(z)) \vee A_F(y) \\ &\leq A_F(z) \vee A_F(y) \end{aligned}$$

by Theorem 3.1. □

Theorem 3.4. *Any ideal of a BCK/BCI-algebra X can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal of X.*

Proof. Let I be an ideal of a BCK/BCI-algebra X and let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X given as follows:

$$A_T : X \to [0,1], \ x \mapsto \begin{cases} \alpha & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$
$$A_I : X \to [0,1], \ x \mapsto \begin{cases} \beta & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$
$$A_F : X \to [0,1], \ x \mapsto \begin{cases} \gamma & \text{if } x \in I, \\ 1 & \text{otherwise} \end{cases}$$

where (α, β, γ) is a fixed ordered triple in $(0,1] \times (0,1] \times [0,1)$. Then $T_\in(A_\sim; \alpha) = I$, $I_\in(A_\sim; \beta) = I$ and $F_\in(A_\sim; \gamma) = I$. Obviously, $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$. Hence

$$\begin{aligned} A_T(x*y) = A_T(y) = A_T(x) = \alpha, \\ A_I(x*y) = A_I(y) = A_I(x) = \beta, \\ A_F(x*y) = A_F(y) = A_F(x) = \gamma, \end{aligned}$$

and so

$$\begin{aligned} A_T(x) \geq A_T(x*y) \wedge A_T(y), \\ A_I(x) \geq A_I(x*y) \wedge A_I(y), \\ A_F(x) \leq A_F(x*y) \vee A_F(y). \end{aligned}$$

If $x * y \notin I$ and $y \notin I$, then

$$\begin{aligned} A_T(x*y) = A_T(y) = 0, \\ A_I(x*y) = A_I(y) = 0, \\ A_F(x*y) = A_F(y) = 1. \end{aligned}$$

Thus

$$\begin{aligned} A_T(x) \geq A_T(x*y) \wedge A_T(y), \\ A_I(x) \geq A_I(x*y) \wedge A_I(y), \\ A_F(x) \leq A_F(x*y) \vee A_F(y). \end{aligned}$$

If $x * y \in I$ and $y \notin I$, then

$$\begin{aligned} A_T(x*y) = \alpha \text{ and } A_T(y) = 0, \\ A_I(x*y) = \beta \text{ and } A_I(y) = 0, \\ A_F(x*y) = \gamma \text{ and } A_F(y) = 1, \end{aligned}$$

It follows that

$$\begin{aligned} A_T(x) \geq 0 = A_T(x*y) \wedge A_T(y), \\ A_I(x) \geq 0 = A_I(x*y) \wedge A_I(y), \\ A_F(x) \leq 1 = A_F(x*y) \vee A_F(y). \end{aligned}$$

Similarly, if $x * y \notin I$ and $y \in I$, then

$$\begin{aligned} A_T(x) \geq A_T(x*y) \wedge A_T(y), \\ A_I(x) \geq A_I(x*y) \wedge A_I(y), \\ A_F(x) \leq A_F(x*y) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. This completes the proof. □

Lemma 3.5 ([5]). *A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \quad (3.5)$$

Theorem 3.6. *In a BCK-algebra, every $(\in, \in)$-neutrosophic ideal is an $(\in, \in)$-neutrosophic subalgebra.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Since $x*y \leq x$ for all $x, y \in X$, it follows from Proposition 3.3 and (3.2) that

$$\begin{aligned} A_T(x*y) \geq A_T(x) \geq A_T(x*y) \wedge A_T(y) \geq A_T(x) \wedge A_T(y), \\ A_I(x*y) \geq A_I(x) \geq A_I(x*y) \wedge A_I(y) \geq A_I(x) \wedge A_I(y), \\ A_F(x*y) \leq A_F(x) \leq A_F(x*y) \vee A_F(y) \leq A_F(x) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X by Lemma 3.5. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Consider a set $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 1.
Then $(X; *, 0)$ is a BCK-algebra (see [6]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2
It is routine to verify that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. We know that $I_\in(A_\sim; \beta)$ is an ideal of X for all $\beta \in (0, 1]$. If $\alpha \in (0.3, 0.7]$, then $T_\in(A_\sim; \alpha) = \{0, 1, 3\}$ is not an ideal of X. Also, if $\gamma \in [0.2, 0.8)$, then $F_\in(A_\sim; \gamma) = \{0, 1, 3\}$ is not an ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.7	0.6	0.2
2	0.3	0.6	0.8
3	0.7	0.4	0.2

We give a condition for an $(\in, \in)$-neutrosophic subalgebra to be an $(\in, \in)$-neutrosophic ideal.

Theorem 3.8. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X that satisfies the condition (3.4), then it is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. Taking $x = y$ in (3.5) and using (III) induce the condition (3.1). Since $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.9. *Let $\{D_k \mid k \in \Lambda^T \cup \Lambda^I \cup \Lambda^F\}$ be a collection of ideals of a BCK/BCI-algebra X, where Λ^T, Λ^I and Λ^F are nonempty subsets of $[0, 1]$, such that*

$$X = \{D_\alpha \mid \alpha \in \Lambda^T\} \cup \{D_\beta \mid \beta \in \Lambda^I\} \cup \{D_\gamma \mid \gamma \in \Lambda^F\}, \tag{3.6}$$

$$(\forall i, j \in \Lambda^T \cup \Lambda^I \cup \Lambda^F)\,(i > j \;\Leftrightarrow\; D_i \subset D_j)\,. \tag{3.7}$$

Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined as follows:

$$\begin{aligned} &A_T : X \to [0,1],\ x \mapsto \bigvee\{\alpha \in \Lambda^T \mid x \in D_\alpha\}, \\ &A_I : X \to [0,1],\ x \mapsto \bigvee\{\beta \in \Lambda^I \mid x \in D_\beta\}, \\ &A_F : X \to [0,1],\ x \mapsto \bigwedge\{\gamma \in \Lambda^F \mid x \in D_\gamma\}. \end{aligned} \tag{3.8}$$

Then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.

Proof. Let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $T_\in(A_\sim; \alpha) \neq \emptyset$, $I_\in(A_\sim; \beta) \neq \emptyset$ and $F_\in(A_\sim; \gamma) \neq \emptyset$. We consider the following two cases:

$$\alpha = \bigvee\{i \in \Lambda^T \mid i < \alpha\} \text{ and } \alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}.$$

First case implies that

$$\begin{aligned} x \in T_\in(A_\sim; \alpha) &\Leftrightarrow x \in D_i \text{ for all } i < \alpha \\ &\Leftrightarrow x \in \cap\{D_i \mid i < \alpha\}. \end{aligned} \tag{3.9}$$

Hence $T_\in(A_\sim; \alpha) = \cap\{D_i \mid i < \alpha\}$, which is an ideal of X. For the second case, we claim that $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$. If $x \in \cup\{D_i \mid i \geq \alpha\}$, then $x \in D_i$ for some $i \geq \alpha$. Thus $A_T(x) \geq i \geq \alpha$, and so $x \in T_\in(A_\sim; \alpha)$. If $x \notin \cup\{D_i \mid i \geq \alpha\}$, then $x \notin D_i$ for all $i \geq \alpha$. Since $\alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda^T = \emptyset$. Hence $x \notin D_i$ for all $i > \alpha - \varepsilon$, which means that if $x \in D_i$ then $i \leq \alpha - \varepsilon$. Thus $A_T(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin T_\in(A_\sim; \alpha)$. Therefore $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$ which is an ideal of X since $\{D_k\}$ forms a chain. Similarly, we can verify that $I_\in(A_\sim; \beta)$ is an ideal of X. Finally, we consider the following two cases:

$$\gamma = \bigwedge\{j \in \Lambda^F \mid \gamma < j\} \text{ and } \gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}.$$

For the first case, we have

$$\begin{aligned} x \in F_\in(A_\sim; \gamma) &\Leftrightarrow x \in D_j \text{ for all } j > \gamma \\ &\Leftrightarrow x \in \cap\{D_j \mid j > \gamma\}, \end{aligned} \tag{3.10}$$

and thus $F_\in(A_\sim; \gamma) = \cap\{D_j \mid j > \gamma\}$ which is an ideal of X. The second case implies that $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$. In fact, if $x \in \cup\{D_j \mid j \leq \gamma\}$, then $x \in D_j$ for some $j \leq \gamma$. Thus $A_F(x) \leq j \leq \gamma$, that is, $x \in F_\in(A_\sim; \gamma)$. Hence $\cup\{D_j \mid j \leq \gamma\} \subseteq F_\in(A_\sim; \gamma)$. Now if $x \notin \cup\{D_j \mid j \leq \gamma\}$, then $x \notin D_j$ for all $j \leq \gamma$. Since $\gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}$, there exists $\varepsilon > 0$ such that $(\gamma, \gamma+\varepsilon) \cap \Lambda^F$ is empty. Hence $x \notin D_j$ for all $j < \gamma + \varepsilon$, and so if $x \in D_j$, then $j \geq \gamma + \varepsilon$. Thus $A_F(x) \geq \gamma + \varepsilon > \gamma$, and hence $x \notin F_\in(A_\sim; \gamma)$. Thus $F_\in(A_\sim; \gamma) \subseteq \cup\{D_j \mid j \leq \gamma\}$, and therefore $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$ which is an ideal of X. Consequently, $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2. □

A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK/BCI-algebras, then $f(0) = 0$. Given a homomorphism $f : X \to Y$ of BCK/BCI-algebras and a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in Y, we define a neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X, which is called the *induced neutrosophic set*, as follows:

$$\begin{aligned} &A^f_T : X \to [0,1],\ x \mapsto A_T(f(x)), \\ &A^f_I : X \to [0,1],\ x \mapsto A_I(f(x)), \\ &A^f_F : X \to [0,1],\ x \mapsto A_F(f(x)). \end{aligned}$$

Theorem 3.10. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y, then the induced neutrosophic set*

$A^f_\sim = (A^f_T, A^f_I, A^f_F)$ *in* X *is an* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. For any $x \in X$, we have

$$A^f_T(x) = A_T(f(x)) \leq A_T(0) = A_T(f(0)) = A^f_T(0),$$
$$A^f_I(x) = A_I(f(x)) \leq A_I(0) = A_I(f(0)) = A^f_I(0),$$
$$A^f_F(x) = A_F(f(x)) \geq A_F(0) = A_F(f(0)) = A^f_F(0).$$

Let $x, y \in X$. Then

$$\begin{aligned} A^f_T(x*y) \wedge A^f_T(y) &= A_T(f(x*y)) \wedge A_T(f(y)) \\ &= A_T(f(x) * f(y)) \wedge A_T(f(y)) \\ &\leq A_T(f(x)) = A^f_T(x), \end{aligned}$$

$$\begin{aligned} A^f_I(x*y) \wedge A^f_I(y) &= A_I(f(x*y)) \wedge A_I(f(y)) \\ &= A_I(f(x) * f(y)) \wedge A_I(f(y)) \\ &\leq A_I(f(x)) = A^f_I(x), \end{aligned}$$

and

$$\begin{aligned} A^f_F(x*y) \vee A^f_F(y) &= A_F(f(x*y)) \vee A_F(f(y)) \\ &= A_F(f(x) * f(y)) \vee A_F(f(y)) \\ &\geq A_F(f(x)) = A^f_F(x). \end{aligned}$$

Therefore $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.11. *Let* $f : X \to Y$ *be an onto homomorphism of* BCK/BCI*-algebras and let* $A_\sim = (A_T, A_I, A_F)$ *be a neutrosophic set in* Y. *If the induced neutrosophic set* $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ *in* X *is an* $(\in, \in)$*-neutrosophic ideal of* X, *then* $A_\sim = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic ideal of* Y.

Proof. Assume that the induced neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$ since f is onto. Using (3.1), we have

$$A_T(x) = A_T(f(a)) = A^f_T(a) \leq A^f_T(0) = A_T(f(0)) = A_T(0),$$
$$A_I(x) = A_I(f(a)) = A^f_I(a) \leq A^f_I(0) = A_I(f(0)) = A_I(0),$$
$$A_F(x) = A_F(f(a)) = A^f_F(a) \geq A^f_F(0) = A_F(f(0)) = A_F(0).$$

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows from (3.2) that

$$\begin{aligned} A_T(x) = A_T(f(a)) &= A^f_T(a) \\ &\geq A^f_T(a*b) \wedge A^f_T(b) \\ &= A_T(f(a*b)) \wedge A_T(f(b)) \\ &= A_T(f(a) * f(b)) \wedge A_T(f(b)) \\ &= A_T(x*y) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) = A_I(f(a)) &= A^f_I(a) \\ &\geq A^f_I(a*b) \wedge A^f_I(b) \\ &= A_I(f(a*b)) \wedge A_I(f(b)) \\ &= A_I(f(a) * f(b)) \wedge A_I(f(b)) \\ &= A_I(x*y) \wedge A_I(y), \end{aligned}$$

and

$$\begin{aligned} A_F(x) = A_F(f(a)) &= A^f_F(a) \\ &\leq A^f_F(a*b) \vee A^f_F(b) \\ &= A_F(f(a*b)) \vee A_F(f(b)) \\ &= A_F(f(a) * f(b)) \vee A_F(f(b)) \\ &= A_F(x*y) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y by Theorem 3.1. □

Let $\mathcal{N}_{(\in,\in)}(X)$ be the collection of all $(\in, \in)$-neutrosophic ideals of X and let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Define binary relations $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned} A_T \mathcal{R}^\alpha_T B_T &\Leftrightarrow T_\in(A_\sim; \alpha) = T_\in(B_\sim; \alpha) \\ A_I \mathcal{R}^\beta_I B_I &\Leftrightarrow I_\in(A_\sim; \beta) = I_\in(B_\sim; \beta) \\ A_F \mathcal{R}^\gamma_F B_F &\Leftrightarrow F_\in(A_\sim; \gamma) = F_\in(B_\sim; \gamma) \end{aligned} \tag{3.11}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$.

Clearly $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ are equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$. For any $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$, let $[A_\sim]_T$ (resp., $[A_\sim]_I$ and $[A_\sim]_F$) denote the equivalence class of $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$ under $\mathcal{R}^\alpha_T$ (resp., $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$). Denote by $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F$ the collection of all equivalence classes under $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$, respectively, that is,

$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T = \{[A_\sim]_T \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),$$
$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I = \{[A_\sim]_I \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),$$
$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F = \{[A_\sim]_F \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X).$$

Now let $\mathcal{I}(X)$ denote the family of all ideals of X. Define maps f_α, g_β and h_γ from $\mathcal{N}_{(\in,\in)}(X)$ to $\mathcal{I}(X) \cup \{\emptyset\}$ by
$f_\alpha(A_\sim) = T_\in(A_\sim; \alpha)$, $g_\beta(A_\sim) = I_\in(A_\sim; \beta)$ and
$h_\gamma(A_\sim) = F_\in(A_\sim; \gamma)$,
respectively, for all $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then f_α, g_β and h_γ are clearly well-defined.

Theorem 3.12. *For any* $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$, *the maps* f_α, g_β *and* h_γ *are surjective from* $\mathcal{N}_{(\in,\in)}(X)$ *to* $\mathcal{I}(X) \cup \{\emptyset\}$.

Proof. Let $0_\sim := (0_T, 0_I, 1_F)$ be a neutrosophic set in X where 0_T, 0_I and 1_F are fuzzy sets in X defined by $0_T(x) = 0$, $0_I(x) = 0$ and $1_F(x) = 1$ for all $x \in X$. Obviously, $0_\sim := (0_T, 0_I, 1_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Also, $f_\alpha(0_\sim) = T_\in(0_\sim; \alpha) = \emptyset$, $g_\beta(0_\sim) = I_\in(0_\sim; \beta) = \emptyset$

and $h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, let $A_\sim = (A_T, A_I, A_F)$ be the $(\in, \in)$-neutrosophic ideal of X in the proof of Theorem 3.4. Then $f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$ and $h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Therefore f_α, g_β and h_γ are surjective. □

Theorem 3.13. *The quotient sets $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$ for any $\alpha,\beta\in(0,1]$ and $\gamma\in[0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$. For any $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$, define

$$\begin{aligned} f_\alpha^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_T \mapsto f_\alpha(A_\sim),\\ g_\beta^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_I \mapsto g_\beta(A_\sim),\\ h_\gamma^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_F \mapsto h_\gamma(A_\sim). \end{aligned}$$

Assume that $f_\alpha(A_\sim) = f_\alpha(B_\sim)$, $g_\beta(A_\sim) = g_\beta(B_\sim)$ and $h_\gamma(A_\sim) = h_\gamma(B_\sim)$ for $B_\sim = (B_T, B_I, B_F) \in \mathcal{N}_{(\in,\in)}(X)$. Then $T_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)$, $I_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)$ and $F_\in(A_\sim;\gamma) = F_\in(B_\sim;\gamma)$ which imply that $A_T\mathcal{R}_T^\alpha B_T$, $A_I\mathcal{R}_I^\beta B_I$ and $A_F\mathcal{R}_F^\gamma B_F$. Hence $[A_\sim]_T = [B_\sim]_T$, $[A_\sim]_I = [B_\sim]_I$ and $[A_\sim]_F = [B_\sim]_F$. Therefore f_α^*, g_β^* and h_γ^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then $f_\alpha^*([0_\sim]_T) = f_\alpha(0_\sim) = T_\in(0_\sim;\alpha) = \emptyset$, $g_\beta^*([0_\sim]_I) = g_\beta(0_\sim) = I_\in(0_\sim;\beta) = \emptyset$, and $h_\gamma^*([0_\sim]_F) = h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then $f_\alpha^*([A_\sim]_T) = f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta^*([A_\sim]_I) = g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$, and $h_\gamma^*([A_\sim]_F) = h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Hence f_α^*, g_β^* and h_γ^* are surjective, and the proof is over. □

For any $\alpha,\beta\in[0,1]$, we define another relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned} (A_\sim,B_\sim)\in\mathcal{R}_\alpha &\Leftrightarrow T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha)\\ &\quad = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha),\\ (A_\sim,B_\sim)\in\mathcal{R}_\beta &\Leftrightarrow I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta)\\ &\quad = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta) \end{aligned} \tag{3.12}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then the relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are also equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$.

Theorem 3.14. *Given $\alpha,\beta\in(0,1)$, we define two maps*

$$\begin{aligned} \varphi_\alpha &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim\mapsto f_\alpha(A_\sim)\cap h_\alpha(A_\sim),\\ \varphi_\beta &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim\mapsto g_\beta(A_\sim)\cap h_\beta(A_\sim) \end{aligned} \tag{3.13}$$

for each $A_\sim = (A_T, A_I, A_F)\in\mathcal{N}_{(\in,\in)}(X)$. Then φ_α and φ_β are surjective.

Proof. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned} \varphi_\alpha(0_\sim) &= f_\alpha(0_\sim)\cap h_\alpha(0_\sim) = T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset,\\ \varphi_\beta(0_\sim) &= g_\beta(0_\sim)\cap h_\beta(0_\sim) = I_\in(0_\sim;\beta)\cap F_\in(0_\sim;\beta) = \emptyset. \end{aligned}$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\begin{aligned} \varphi_\alpha(A_\sim) &= f_\alpha(A_\sim)\cap h_\alpha(A_\sim)\\ &= T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = I \end{aligned}$$

and

$$\begin{aligned} \varphi_\beta(A_\sim) &= g_\beta(A_\sim)\cap h_\beta(A_\sim)\\ &= I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I. \end{aligned}$$

Therefore φ_α and φ_β are surjective. □

Theorem 3.15. *For any $\alpha,\beta\in(0,1)$, the quotient sets $\mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $\mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$.*

Proof. Given $\alpha,\beta\in(0,1)$, define two maps φ_α^* and φ_β^* as follows:

$$\begin{aligned} \varphi_\alpha^* &: \mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\alpha}\mapsto\varphi_\alpha(A_\sim),\\ \varphi_\beta^* &: \mathcal{N}_{(\in,\in)}(X)/\varphi_\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\beta}\mapsto\varphi_\beta(A_\sim). \end{aligned}$$

If $\varphi_\alpha^*([A_\sim]_{\mathcal{R}_\alpha}) = \varphi_\alpha^*([B_\sim]_{\mathcal{R}_\alpha})$ and $\varphi_\beta^*([A_\sim]_{\mathcal{R}_\beta}) = \varphi_\beta^*([B_\sim]_{\mathcal{R}_\beta})$ for all $[A_\sim]_{\mathcal{R}_\alpha}, [B_\sim]_{\mathcal{R}_\alpha}\in\mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $[A_\sim]_{\mathcal{R}_\beta}, [B_\sim]_{\mathcal{R}_\beta}\in\mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$, then

$$f_\alpha(A_\sim)\cap h_\alpha(A_\sim) = f_\alpha(B_\sim)\cap h_\alpha(B_\sim)$$

and

$$g_\beta(A_\sim)\cap h_\beta(A_\sim) = g_\beta(B_\sim)\cap h_\beta(B_\sim),$$

that is,

$$T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha)$$

and

$$I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta).$$

Hence $(A_\sim,B_\sim)\in\mathcal{R}_\alpha$ and $(A_\sim,B_\sim)\in\mathcal{R}_\beta$. It follows that $[A_\sim]_{\mathcal{R}_\alpha} = [B_\sim]_{\mathcal{R}_\alpha}$ and $[A_\sim]_{\mathcal{R}_\beta} = [B_\sim]_{\mathcal{R}_\beta}$. Thus φ_α^* and φ_β^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned} \varphi_\alpha^*([0_\sim]_{\mathcal{R}_\alpha}) &= \varphi_\alpha(0_\sim) = f_\alpha(0_\sim)\cap h_\alpha(0_\sim)\\ &= T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset \end{aligned}$$

and

$$\varphi_\beta^*\left([0_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(0_\sim) = g_\beta(0_\sim) \cap h_\beta(0_\sim) = I_\in(0_\sim;\beta) \cap F_\in(0_\sim;\beta) = \emptyset.$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\varphi_\alpha^*\left([A_\sim]_{\mathcal{R}_\alpha}\right) = \varphi_\alpha(A_\sim) = f_\alpha(A_\sim) \cap h_\alpha(A_\sim) = T_\in(A_\sim;\alpha) \cap F_\in(A_\sim;\alpha) = I$$

and

$$\varphi_\beta^*\left([A_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(A_\sim) = g_\beta(A_\sim) \cap h_\beta(A_\sim) = I_\in(A_\sim;\beta) \cap F_\in(A_\sim;\beta) = I.$$

Therefore φ_α^* and φ_β^* are surjective. This completes the proof. □

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Commutative falling neutrosophic ideals in BCK-algebras

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Abstract: The notions of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a commutative $(\in, \in)$-neutrosophic ideal are obtained. Relations between commutative $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal are established. Relations between commutative $(\in, \in)$-neutrosophic ideal, falling neutrosophic ideal and commutative falling neutrosophic ideal are considered. Conditions for a falling neutrosophic ideal to be commutative are provided.

Keywords: (commutative) $(\in, \in)$-neutrosophic ideal; neutrosophic random set; neutrosophic falling shadow; (commutative) falling neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [11, 12, 13] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [6] and [10]). Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [16] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [17]. Tan et al. [14, 15] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [7] considered a fuzzy subalgebra and a fuzzy ideal as the falling shadow of the cloud of the subalgebra and ideal. Jun et al. [8] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal.

In this paper, we introduce the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a commutative $(\in, \in)$-neutrosophic ideal, and discuss relations between a commutative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal, and consider relations between a commutative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be commutative.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [3] and [4]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X

satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} x \le y \;\Rightarrow\; x * z \le y * z \\ x \le y \;\Rightarrow\; z * y \le z * x \end{array}\right), \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (2.5) and

$$(x * y) * z \in I,\ z \in I \;\Rightarrow\; x * (y * (y * x)) \in I \tag{2.7}$$

for all $x, y, z \in X$.

Observe that every commutative ideal is an ideal, but the converse is not true (see [9]).

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [12]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_\in(A; \alpha) := \{x \in X \mid A_T(x) \ge \alpha\},$$
$$I_\in(A; \beta) := \{x \in X \mid A_I(x) \ge \beta\},$$
$$F_\in(A; \gamma) := \{x \in X \mid A_F(x) \le \gamma\}.$$

We say $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are *neutrosophic* $\in$*-subsets*.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic subalgebra* of X (see [6]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A; \alpha_x),\ y \in T_\in(A; \alpha_y) \\ \qquad \Rightarrow\ x * y \in T_\in(A; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A; \beta_x),\ y \in I_\in(A; \beta_y) \\ \qquad \Rightarrow\ x * y \in I_\in(A; \beta_x \wedge \beta_y), \\ x \in F_\in(A; \gamma_x),\ y \in F_\in(A; \gamma_y) \\ \qquad \Rightarrow\ x * y \in F_\in(A; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.8}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic ideal* of X (see [10]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A; \alpha_x) \;\Rightarrow\; 0 \in T_\in(A; \alpha_x) \\ x \in I_\in(A; \beta_x) \;\Rightarrow\; 0 \in I_\in(A; \beta_x) \\ x \in F_\in(A; \gamma_x) \;\Rightarrow\; 0 \in F_\in(A; \gamma_x) \end{array}\right) \tag{2.9}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A; \alpha_x),\ y \in T_\in(A; \alpha_y) \\ \qquad \Rightarrow\ x \in T_\in(A; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A; \beta_x),\ y \in I_\in(A; \beta_y) \\ \qquad \Rightarrow\ x \in I_\in(A; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A; \gamma_x),\ y \in F_\in(A; \gamma_y) \\ \qquad \Rightarrow\ x \in F_\in(A; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.10}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.11}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.12}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic random set* on X (see [8]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables,

that is,

$$(\forall C \in \mathcal{B}) \begin{pmatrix} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{pmatrix}. \tag{2.13}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)),$$
$$\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)),$$
$$\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)).$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [8]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma)$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [8]).

3 Commutative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X is called a *commutative* $(\in, \in)$*-neutrosophic ideal* of X if it satisfies the condition (2.9) and

$$\begin{aligned} &(x * y) * z \in T_\in(A; \alpha_x),\ z \in T_\in(A; \alpha_y) \\ &\qquad \Rightarrow\ x * (y * (y * x)) \in T_\in(A; \alpha_x \wedge \alpha_y) \\ &(x * y) * z \in I_\in(A; \beta_x),\ z \in I_\in(A; \beta_y) \\ &\qquad \Rightarrow\ x * (y * (y * x)) \in I_\in(A; \beta_x \wedge \beta_y) \\ &(x * y) * z \in F_\in(A; \gamma_x),\ z \in F_\in(A; \gamma_y) \\ &\qquad \Rightarrow\ x * (y * (y * x)) \in F_\in(A; \gamma_x \vee \gamma_y) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.2. Consider a set $X = \{0,1,2,3\}$ with the binary operation $*$ which is given in Table 1.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.3	0.6	0.8
2	0.3	0.6	0.8
3	0.5	0.4	0.7

It is routine to verify that $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{pmatrix} \tag{3.2}$$

and for all $x, y, z \in X$,

$$\begin{aligned} &A_T(x * (y * (y * x))) \\ &\qquad \geq A_T((x * y) * z) \wedge A_T(z) \\ &A_I(x * (y * (y * x))) \\ &\qquad \geq A_I((x * y) * z) \wedge A_I(z) \\ &A_F(x * (y * (y * x))) \\ &\qquad \leq A_F((x * y) * z) \vee A_F(z) \end{aligned} \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A; A_T(a))$ and $0 \notin T_\in(A; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly,

$A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_{\in}(A; A_F(a))$ and $0 \notin F_{\in}(A; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a * (b * (b * a))) < A_T((a * b) * c) \wedge A_T(c).$$

Taking $\alpha := A_T((a * b) * c) \wedge A_T(c)$ implies that $(a * b) * c \in T_{\in}(A; \alpha)$ and $c \in T_{\in}(A; \alpha)$ but $a * (b * (b * a)) \notin T_{\in}(A; \alpha)$, which is a contradiction. Hence

$$A_T(x * (y * (y * x))) \geq A_T((x * y) * z) \wedge A_T(z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x * (y * (y * x))) > A_F((x * y) * z) \vee A_F(z) := \gamma.$$

Then $(x*y)*z \in F_{\in}(A; \gamma)$ and $z \in F_{\in}(A; \gamma)$ but $x*(y*(y*x)) \notin F_{\in}(A; \gamma)$, a contradiction. Thus

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z)$$

for all $x, y, z \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Let $x \in T_{\in}(A; \alpha)$, $a \in I_{\in}(A; \beta)$ and $u \in F_{\in}(A; \gamma)$ for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_{\in}(A; \alpha)$, $0 \in I_{\in}(A; \beta)$ and $0 \in F_{\in}(A; \gamma)$. Let $a, b, c \in X$ be such that $(a * b) * c \in T_{\in}(A; \alpha)$ and $c \in T_{\in}(A; \alpha)$ for $\alpha \in (0, 1]$. Then

$$A_T(a * (b * (b * a))) \geq A_T((a * b) * c) \wedge A_T(c) \geq \alpha$$

by (3.3), and so $a * (b * (b * a)) \in T_{\in}(A; \alpha)$. If $(x * y) * z \in I_{\in}(A; \beta)$ and $z \in I_{\in}(A; \beta)$ for all $x, y, z \in X$ and $\beta \in (0, 1]$, then $A_I((x * y) * z) \geq \beta$ and $A_I(z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z) \geq \beta,$$

that is, $x * (y * (y * x)) \in I_{\in}(A; \beta)$. Finally, suppose that

$$(x * y) * z \in F_{\in}(A; \gamma) \text{ and } z \in F_{\in}(A; \gamma)$$

for all $x, y, z \in X$ and $\gamma \in (0, 1]$. Then $A_F((x * y) * z) \leq \gamma$ and $A_F(z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z) \leq \gamma.$$

Hence $x * (y * (y * x)) \in F_{\in}(A; \gamma)$. Therefore the non-empty $\in$-subsets $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. □

Theorem 3.4. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X and assume that $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then there exist $x, y, z \in X$ such that $x \in T_{\in}(A; \alpha)$, $y \in I_{\in}(A; \beta)$ and $z \in F_{\in}(A; \gamma)$. It follows from (2.9) that $0 \in T_{\in}(A; \alpha)$, $0 \in I_{\in}(A; \beta)$ and $0 \in F_{\in}(A; \gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that

$$\begin{aligned}
&(x * y) * z \in T_{\in}(A; \alpha),\ z \in T_{\in}(A; \alpha),\\
&(a * b) * c \in I_{\in}(A; \beta),\ c \in I_{\in}(A; \beta),\\
&(u * v) * w \in F_{\in}(A; \gamma),\ w \in F_{\in}(A; \gamma).
\end{aligned}$$

Then

$$\begin{aligned}
x * (y * (y * x)) &\in T_{\in}(A; \alpha \wedge \alpha) = T_{\in}(A; \alpha),\\
a * (b * (b * a)) &\in I_{\in}(A; \beta \wedge \beta) = I_{\in}(A; \beta),\\
u * (v * (v * u)) &\in F_{\in}(A; \gamma \vee \gamma) = F_{\in}(A; \gamma)
\end{aligned}$$

by (2.10). Hence the non-empty neutrosophic $\in$-subsets $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are nonempty and are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Obviously, (2.9) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0, 1]$ be such that $(x * y) * z \in T_{\in}(A; \alpha_x)$ and $z \in T_{\in}(A; \alpha_y)$. Then $(x * y) * z \in T_{\in}(A; \alpha)$ and $z \in T_{\in}(A; \alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_{\in}(A; \alpha)$ is a commutative ideal of X, it follows that

$$x * (y * (y * x)) \in T_{\in}(A; \alpha) = T_{\in}(A; \alpha_x \wedge \alpha_y).$$

Similarly, if $(x * y) * z \in I_{\in}(A; \beta_x)$ and $z \in I_{\in}(A; \beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0, 1]$, then

$$x * (y * (y * x)) \in I_{\in}(A; \beta_x \wedge \beta_y).$$

Now, suppose that $(x*y)*z \in F_{\in}(A; \gamma_x)$ and $z \in F_{\in}(A; \gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0, 1)$. Then $(x * y) * z \in F_{\in}(A; \gamma)$ and $z \in F_{\in}(A; \gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence

$$x * (y * (y * x)) \in F_{\in}(A; \gamma) = F_{\in}(A; \gamma_x \vee \gamma_y)$$

since $F_{\in}(A; \gamma)$ is a commutative ideal of X. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.5. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commuta-*

tive $(\in, \in)$*-neutrosophic ideal of* X *if and only if it satisfies two conditions* (3.2) *and* (3.3).

Proposition 3.6. *Every commutative* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$(\forall x, y \in X)\begin{pmatrix} x * y \in T_{\in}(A;\alpha) \\ \quad \Rightarrow x * (y * (y * x)) \in T_{\in}(A;\alpha) \\ x * y \in I_{\in}(A;\beta) \\ \quad \Rightarrow x * (y * (y * x)) \in I_{\in}(A;\beta) \\ x * y \in F_{\in}(A;\gamma) \\ \quad \Rightarrow x * (y * (y * x)) \in F_{\in}(A;\gamma) \end{pmatrix} \tag{3.4}$$

for all $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$.

Proof. It is induced by taking $z = 0$ in (3.1). □

Theorem 3.7. *Every commutative* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *is an* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Assume that

$$\begin{aligned} &x * y \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y), \\ &a * b \in I_{\in}(A;\beta_a),\ b \in I_{\in}(A;\beta_b), \\ &c * d \in F_{\in}(A;\gamma_c),\ d \in F_{\in}(A;\gamma_d) \end{aligned}$$

for all $x, y, a, b, c, d \in X$. Using (2.1), we have

$$\begin{aligned} &(x * 0) * y = x * y \in T_{\in}(A;\alpha_x), \\ &(a * 0) * b = a * b \in I_{\in}(A;\beta_a), \\ &(c * 0) * d = c * d \in F_{\in}(A;\gamma_c). \end{aligned}$$

It follows from (3.1), (2.1) and (V) that

$$\begin{aligned} &x = x * 0 = x * (0 * (0 * x)) \in T_{\in}(A;\alpha_x \wedge \alpha_y), \\ &a = a * 0 = a * (0 * (0 * a)) \in I_{\in}(A;\beta_a \wedge \beta_b), \\ &c = c * 0 = c * (0 * (0 * c)) \in F_{\in}(A;\gamma_c \vee \gamma_d). \end{aligned}$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4

Table 4: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.66	0.77	0.27
1	0.55	0.45	0.37
2	0.33	0.66	0.47
3	0.33	0.45	0.67
4	0.33	0.45	0.67

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. But it is not a commutative $(\in, \in)$-neutrosophic ideal of X since $(2 * 3) * 0 \in T_{\in}(A; 0.6)$ and $0 \in T_{\in}(A; 0.5)$ but $2 * (3 * (3 * 2)) \notin T_{\in}(A; 0.5 \wedge 0.6)$, $(1 * 3) * 2 \in I_{\in}(A; 0.55)$ and $2 \in I_{\in}(A; 0.63)$ but $1 * (3 * (3 * 1)) \notin I_{\in}(A; 0.55 \wedge 0.63)$, and/or $(2 * 3) * 0 \in F_{\in}(A; 0.43)$ and $0 \in F_{\in}(A; 0.39)$ but $2 * (3 * (3 * 2)) \notin F_{\in}(A; 0.43 \vee 0.39)$.

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal.

Theorem 3.9. *Let* $A = (A_T, A_I, A_F)$ *be an* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *in which the condition* (3.4) *is valid. Then* $A = (A_T, A_I, A_F)$ *is a commutative* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X and $x, y, z \in X$ be such that $(x * y) * z \in T_{\in}(A; \alpha_x)$ and $z \in T_{\in}(A; \alpha_y)$ for $\alpha_x, \alpha_y \in (0, 1]$. Then $x * y \in T_{\in}(A; \alpha_x \wedge \alpha_y)$ since $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. It follows from (3.4) that $x * (y * (y * x)) \in T_{\in}(A; \alpha_x \wedge \alpha_y)$. Similarly, if $(x * y) * z \in I_{\in}(A; \beta_x)$ and $z \in I_{\in}(A; \beta_y)$, then $x * (y * (y * x)) \in I_{\in}(A; \beta_x \wedge \beta_y)$. Let $a, b, c \in X$ and $\gamma_a, \gamma_b \in [0, 1)$ be such that $(a * b) * c \in F_{\in}(A; \gamma_a)$ and $c \in F_{\in}(A; \gamma_a)$. Then $a * b \in F_{\in}(A; \gamma_a \vee \gamma_b)$, which implies from (3.4) that $a * (b * (b * a)) \in F_{\in}(A; \gamma_a \vee \gamma_b)$. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Lemma 3.10. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$\begin{aligned} &y, z \in T_{\in}(A;\alpha) \ \Rightarrow\ x \in T_{\in}(A;\alpha) \\ &y, z \in I_{\in}(A;\beta) \ \Rightarrow\ x \in I_{\in}(A;\beta) \\ &y, z \in F_{\in}(A;\gamma) \ \Rightarrow\ x \in F_{\in}(A;\gamma) \end{aligned} \tag{3.5}$$

for all $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ *and* $x, y, z \in X$ *with* $x * y \leq z$.

Proof. For any $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ and $x, y, z \in X$ with $x * y \leq z$, let $y, z \in T_{\in}(A; \alpha)$, $y, z \in I_{\in}(A; \beta)$ and $y, z \in F_{\in}(A; \gamma)$. Then

$$(x * y) * z = 0 \in T_{\in}(A;\alpha) \cap I_{\in}(A;\beta) \cap F_{\in}(A;\gamma)$$

by (2.9). It follows from (2.10) that

$$x * y \in T_{\in}(A;\alpha) \cap I_{\in}(A;\beta) \cap F_{\in}(A;\gamma)$$

and so that

$$x \in T_{\in}(A;\alpha) \cap I_{\in}(A;\beta) \cap F_{\in}(A;\gamma).$$

Thus (3.5) is valid. □

Theorem 3.11. *In a commutative BCK-algebra, every $(\in, \in)$-neutrosophic ideal is a commutative $(\in, \in)$-neutrosophic ideal.*

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a commutative BCK-algebra X. Let $x, y, z \in X$ be such that

$$(x * y) * z \in T_{\in}(A;\alpha_x) \cap I_{\in}(A;\beta_x) \cap F_{\in}(A;\gamma_x)$$

and

$$z \in T_{\in}(A;\alpha_y) \cap I_{\in}(A;\beta_y) \cap F_{\in}(A;\gamma_y)$$

for $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$. Note that

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\leq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) \\
&= 0
\end{aligned}$$

by (2.3), (2.4) and (III), which implies that

$$(x * (y * (y * x))) * ((x * y) * z) \leq z.$$

It follows from Lemma 3.10 that

$$x * (y * (y * x)) \in T_{\in}(A;\alpha_x) \cap I_{\in}(A;\beta_x) \cap F_{\in}(A;\gamma_x).$$

Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

4 Commutative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *commutative falling neutrosophic ideal* of X if $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$.

Example 4.2. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 5
Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,4\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,3,4\} & \text{if } t \in [0.85, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,4\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{1, 2\}, \\ 0.4 & \text{if } x = 3, \\ 0.45 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x \in \{0, 1, 2\}, \\ 0.3 & \text{if } x = 3, \\ 0.25 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \in \{1, 2, 4\}, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set

$\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\begin{aligned}\Omega(x;\xi_T) &:= \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},\\ \Omega(x;\xi_I) &:= \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},\\ \Omega(x;\xi_F) &:= \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.\end{aligned}$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [8]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then*

$$\begin{aligned}&\Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_T)\\ &\Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_I)\\ &\Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_F)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\Omega(x*(y*(y*x));\xi_T) \subseteq \Omega((x*y)*z;\xi_T)\\ &\Omega(x*(y*(y*x));\xi_I) \subseteq \Omega((x*y)*z;\xi_I)\\ &\Omega(x*(y*(y*x));\xi_F) \subseteq \Omega((x*y)*z;\xi_F)\end{aligned} \tag{4.2}$$

for all $x, y, z \in X$.

Proof. Let

$$\begin{aligned}&\omega_T \in \Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T),\\ &\omega_I \in \Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I),\\ &\omega_F \in \Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F)\end{aligned}$$

for all $x, y, z \in X$. Then

$$\begin{aligned}&(x*y)*z \in \xi_T(\omega_T) \text{ and } z \in \xi_T(\omega_T),\\ &(x*y)*z \in \xi_I(\omega_I) \text{ and } z \in \xi_I(\omega_I),\\ &(x*y)*z \in \xi_F(\omega_F) \text{ and } z \in \xi_F(\omega_F).\end{aligned}$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X, it follows from (2.7) that

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$$

and so that

$$\begin{aligned}&\omega_T \in \Omega(x*(y*(y*x));\xi_T),\\ &\omega_I \in \Omega(x*(y*(y*x));\xi_I),\\ &\omega_F \in \Omega(x*(y*(y*x));\xi_F).\end{aligned}$$

Hence (4.1) is valid. Now let

$$\begin{aligned}&\omega_T \in \Omega(x*(y*(y*x));\xi_T),\\ &\omega_I \in \Omega(x*(y*(y*x));\xi_I),\\ &\omega_F \in \Omega(x*(y*(y*x));\xi_F)\end{aligned}$$

for all $x, y, z \in X$. Then

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Note that

$$\begin{aligned}&((x*y)*z)*(x*(y*(y*x)))\\ &= ((x*y)*(x*(y*(y*x))))*z\\ &\le ((y*(y*x))*y)*z = ((y*y)*(y*x))*z\\ &= (0*(y*x))*z = 0*z = 0,\end{aligned}$$

which yields

$$\begin{aligned}&((x*y)*z)*(x*(y*(y*x)))\\ &= 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).\end{aligned}$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals and hence ideals of X, it follows that

$$(x*y)*z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Hence

$$\begin{aligned}&\omega_T \in \Omega((x*y)*z;\xi_T),\\ &\omega_I \in \Omega((x*y)*z;\xi_I),\\ &\omega_F \in \Omega((x*y)*z;\xi_F).\end{aligned}$$

Therefore (4.2) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.3}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.4}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [7]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$\begin{aligned}A^g_T &:= \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},\\ A^g_I &:= \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},\\ A^g_F &:= \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}\end{aligned}$$

and

$$\begin{aligned}&\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},\\ &\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},\\ &\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.\end{aligned}$$

Then $A^g_T, A^g_I, A^g_F \in \mathcal{A}$ (see [8]).

Theorem 4.4. *If K is a commutative ideal of a BCK-algebra X, then*

$$\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},$$
$$\xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},$$
$$\xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}$$

are commutative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a commutative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have $\theta \in \xi_T(\omega_T)$, $\theta \in \xi_I(\omega_I)$ and $\theta \in \xi_F(\omega_F)$. Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that

$$(f_T \circledast g_T) \circledast h_T \in \xi_T(\omega_T) \text{ and } h_T \in \xi_T(\omega_T).$$

Then

$$(f_T(\omega_T) * g_T(\omega_T)) * h_T(\omega_T) = ((f_T \circledast g_T) \circledast h_T)(\omega_T) \in K$$

and $h_T(\omega_T) \in K$. Since K is a commutative ideal of X, it follows from (2.7) that

$$(f_T \circledast (g_T \circledast (g_T \circledast f_T)))(\omega_T)$$
$$= f_T(\omega_T) * (g_T(\omega_T) * (g_T(\omega_T) * f_T(\omega_T))) \in K,$$

that is, $f_T \circledast (g_T \circledast (g_T \circledast f_T)) \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is a commutative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is a commutative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast g_F) \circledast h_F \in \xi_F(\omega_F)$ and $h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * g_F(\omega_F)) * h_F(\omega_F)$$
$$= ((f_F \circledast g_F) \circledast h_F)(\omega_F) \in K$$

and $h_F(\omega_F) \in K$. Then

$$(f_F \circledast (g_F \circledast (g_F \circledast f_F)))(\omega_F)$$
$$= f_F(\omega_F) * (g_F(\omega_F) * (g_F(\omega_F) * f_F(\omega_F))) \in K,$$

and so $f_F \circledast (g_F \circledast (g_F \circledast f_F)) \in \xi_F(\omega_F)$. Hence $\xi_F(\omega_F)$ is a commutative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is a commutative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X. Then $T_\in(\tilde{H}; \alpha)$, $I_\in(\tilde{H}; \beta)$ and $F_\in(\tilde{H}; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H}; \alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H}; \beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H}; \gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X. □

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 6

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.2), \\ \{0,2\} & \text{if } t \in [0.2, 0.55), \\ \{0,2,4\} & \text{if } t \in [0.55, 0.75), \\ \{0,1,2,3\} & \text{if } t \in [0.75, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.34), \\ \{0,4\} & \text{if } t \in [0.34, 0.66), \\ \{0,1,4\} & \text{if } t \in [0.66, 0.78), \\ X & \text{if } t \in [0.78, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.87, 1], \\ \{0,2\} & \text{if } t \in (0.76, 0.87], \\ \{0,4\} & \text{if } t \in (0.58, 0.76], \\ \{0,2,4\} & \text{if } t \in (0.33, 0.58], \\ X & \text{if } t \in [0, 0.33]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.68 & \text{if } x = 1, \\ 0.22 & \text{if } x \in \{2,3\}, \\ 0.66 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.67 & \text{if } x \in \{1, 3\}, \\ 0.31 & \text{if } x = 2, \\ 0.24 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not a commutative $(\in, \in)$-neutrosophic ideal of X since

$$(3 * 4) * 2 \in T_{\in}(\tilde{H}; 0.4) \text{ and } 2 \in T_{\in}(\tilde{H}; 0.6),$$

but $3 * (4 * (4 * 3)) = 3 \notin T_{\in}(\tilde{H}; 0.4)$.

We provide relations between a falling neutrosophic ideal and a commutative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all ω_T, ω_I, $\omega_F \in \Omega$. Thus $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all ω_T, ω_I, $\omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 3\} & \text{if } t \in [0, 0.27), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.27, 0.66), \\ \{0, 1, 2, 4\} & \text{if } t \in [0.67, 1], \end{cases}$$

$$\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 3\} & \text{if } t \in [0, 0.35), \\ \{0, 1, 2, 4\} & \text{if } t \in [0.35, 1], \end{cases}$$

and

$$\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.84, 1], \\ \{0, 3\} & \text{if } t \in (0.76, 0.84], \\ \{0, 1, 2, 4\} & \text{if } t \in (0.58, 0.76], \\ X & \text{if } t \in [0, 0.58]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X. But it is not a commutative falling neutrosophic ideal of X because if $\alpha \in [0, 0.27)$, $\beta \in [0, 0.35)$ and $\gamma \in (0.76, 0.84]$, then $\xi_T(\alpha) = \{0, 3\}$, $\xi_I(\beta) = \{0, 3\}$ and $\xi_F(\gamma) = \{0, 3\}$ are not commutative ideals of X respectively.

Since every ideal is commutative in a commutative BCK-algebra, we have the following theorem.

Theorem 4.9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a commutative BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$(\forall x, y \in X)(x \le y \ \Rightarrow \ x \le y * (y * x)), \tag{4.5}$$
$$(\forall x, y \in X)(x \le y \ \Rightarrow \ x = y * (y * x)), \tag{4.6}$$
$$(\forall x, y \in X)(x * (x * y) = y * (y * (x * (x * y)))), \tag{4.7}$$
$$(\forall x, y, z \in X)(x, y \le z, z * y \le z * x \ \Rightarrow \ x \le y), \tag{4.8}$$
$$(\forall x, y, z \in X)(x \le z, z * y \le z * x \ \Rightarrow \ x \le y), \tag{4.9}$$

let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.

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On Neutrosophic Soft Prime Ideal

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Abstract The motivation of the present paper is to extend the concept of neutrosophic soft prime ideal over a ring. In this paper the concept of neutrosophic soft completely prime ideals, neutrosophic soft completely semi-prime ideals and neutrosophic soft prime k - ideals have been introduced. These are illustrated with suitable examples also. Several related properties, theorems and structural characteristics of each are studied here.

Keywords Neutrosophic soft completely prime ideals; Neutrosophic soft completely semi-prime ideals; Neutrosophic soft prime k - ideals.

1 Introduction

Because of the insufficiency in the available information situation, evaluation of membership values and nonmembership values are not always possible to handle the uncertainties appearing in daily life situations. So there exists an indeterministic part upon which hesitation survives. The neutrosophic set theory by Smarandache [1,2] which is a generalisation of fuzzy set and intuitionistic fuzzy set theory, makes description of the objective world more realistic, practical and very promising in nature. The neutrosophic logic includes the information about the percentage of truth, indeterminacy and falsity grade in several real world problems in law, medicine, engineering, management, industrial, IT sector etc which are not available in intuitionistic fuzzy set theory. But each of the theories suffers from inherent difficulties because of the inadequacy of parametrization tools. Molodtsov [3] introduced a nice concept of soft set theory which is free from the parametrization inadequacy syndrome of different theories dealing with uncertainty. The parametrization tool of soft set theory makes it very convenient and easy to apply in practice. The classical algebraic structures were extended over fuzzy set, intuitionistic fuzzy set, soft set, fuzzy soft set and intuitionistic fuzzy soft set by so many authors, for instance, Rosenfeld [4], Malik and

Mordeson [5,6], Lavanya and Kumar [8], Bakhadach et al. [9], Dutta et al. [10-12], Maji et al. [13], Aktas and Cagman [14], Augunoglu and Aygun [15], Zhang [16], Maheswari and Meera [17] and others.

The notion of neutrosophic soft set theory (NSS) has been innovated by Maji [18]. Later, it has been modified by Deli and Broumi [19]. Cetkin et al. [20,21], Bera and Mahapatra [22-26] and others have produced their research works on fundamental algebraic structures on the NSS theory context.

This paper presents the notion of neutrosophic soft completely prime ideals, neutrosophic soft completely semi-prime ideals and neutrosophic soft prime k-ideals along with investigation of some related properties and theorems. The content of the present paper is designed as following :

Section 2 gives some preliminary useful definitions related to it. In Section 3, neutrosophic soft completely prime ideals is defined and illustrated by suitable examples along with investigation of its structural characteristics. Section 4 deals with the notion of neutrosophic soft completely semi-prime ideals with development of related theorems. The concept of neutrosophic soft prime k-ideals along with some properties has been introduced in Section 6. Finally, the conclusion of our work has been stated in Section 7.

2 Preliminaries

We recall some basic definitions related to fuzzy set, soft set, neutrosophic soft set for the sake of completeness.

2.1 Definition [24]

1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be continuous t - norm if $*$ satisfies the following conditions :
(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a$, $\forall a \in [0,1]$.
(iv) $a * b \leq c * d$ if $a \leq c$, $b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be continuous t - conorm (s - norm) if $\diamond$ satisfies the following conditions :
(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a$, $\forall a \in [0,1]$.
(iv) $a \diamond b \leq c \diamond d$ if $a \leq c$, $b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

2.2 Definition [1]

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is $T_A, I_A, F_A : X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$, $F_A(x)$ and so, $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

2.3 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.4 Definition [18]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to NS(U)$ is a mapping.

This concept has been redefined by Deli and Broumi [19] as given below.

2.5 Definition [19]

1. Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,

$$\begin{aligned} N &= \{(e, f_N(e)) : e \in E\} \\ &= \{(e, \{< x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) >: x \in U\}) : e \in E\} \end{aligned}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if $T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x)$, $I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x)$, $F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x)$, $\forall e \in E$ and $\forall x \in U$.

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

2.6 Proposition [22]

An NSS N over the group (G, o) is called a neutrosophic soft group iff followings hold on the assumption that $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$.

$$\begin{aligned} T_{f_N(e)}(xoy^{-1}) &\geq T_{f_N(e)}(x) * T_{f_N(e)}(y), \\ I_{f_N(e)}(xoy^{-1}) &\leq I_{f_N(e)}(x) \diamond I_{f_N(e)}(y), \\ F_{f_N(e)}(xoy^{-1}) &\leq F_{f_N(e)}(x) \diamond F_{f_N(e)}(y)); \ \forall x, y \in G, \forall e \in E. \end{aligned}$$

2.7 Definition [24]

1. A neutrosophic soft ring N over the ring $(R, +, \cdot)$ is called a neutrosophic soft left ideal over R if $f_N(e)$ is a neutrosophic left ideal of R for each $e \in E$ i.e.,
(i) $f_N(e)$ is a neutrosophic subgroup of $(R, +)$ for each $e \in E$ and

$$(ii) \begin{cases} T_{f_N(e)}(x \cdot y) \geq T_{f_N(e)}(y) \\ I_{f_N(e)}(x \cdot y) \leq I_{f_N(e)}(y) \\ F_{f_N(e)}(x \cdot y) \leq F_{f_N(e)}(y); \text{ for } x, y \in R. \end{cases}$$

2. A neutrosophic soft ring N over the ring $(R, +, \cdot)$ is called a neutrosophic soft right ideal over R if $f_N(e)$ is a neutrosophic right ideal of R for each $e \in E$ i.e.,
(i) $f_N(e)$ is a neutrosophic subgroup of $(R, +)$ for each $e \in E$ and

$$(ii) \begin{cases} T_{f_N(e)}(x \cdot y) \geq T_{f_N(e)}(x) \\ I_{f_N(e)}(x \cdot y) \leq I_{f_N(e)}(x) \\ F_{f_N(e)}(x \cdot y) \leq F_{f_N(e)}(x); \text{ for } x, y \in R. \end{cases}$$

3. A neutrosophic soft ring N over the ring $(R, +, \cdot)$ is called a neutrosophic soft ideal over R if $f_N(e)$ is a both neutrosophic left and right ideal of R for each $e \in E$.

2.8 Definition [25]

1. Let $\varphi : U \to V$ and $\psi : E \to E$ be two functions where E is the parameter set for each of the crisp sets U and V. Then the pair (φ, ψ) is called an NSS function from (U, E) to (V, E). We write, $(\varphi, \psi) : (U, E) \to (V, E)$. If M is an NSS over U via parametric set E, we shall write (M, E) an NSS over U.

2. Let (M, E), (N, E) be two NSSs defined over U, V respectively and (φ, ψ) be an NSS function from (U, E) to (V, E). Then,
(i) The image of (M, E) under (φ, ψ), denoted by $(\varphi, \psi)(M, E)$, is an NSS over V and is defined by :
$(\varphi, \psi)(M, E) = (\varphi(M), \psi(E)) = \{< \psi(a), f_{\varphi(M)} >: a \in E\}$ where $\forall b \in \psi(E), \forall y \in V$,

$$T_{f_{\varphi(M)}(b)}(y) = \begin{cases} \max_{\varphi(x)=y} \max_{\psi(a)=b} [T_{f_M(a)}(x)], & \text{if } x \in \varphi^{-1}(y) \\ 0 \quad , \quad \text{otherwise.} \end{cases}$$

$$I_{f_{\varphi(M)}(b)}(y) = \begin{cases} \min_{\varphi(x)=y} \min_{\psi(a)=b} [I_{f_M(a)}(x)], & \text{if } x \in \varphi^{-1}(y) \\ 1 \quad , \quad \text{otherwise.} \end{cases}$$

$$F_{f_{\varphi(M)}(b)}(y) = \begin{cases} \min_{\varphi(x)=y} \min_{\psi(a)=b} [F_{f_M(a)}(x)], & \text{if } x \in \varphi^{-1}(y) \\ 1 \quad , \quad \text{otherwise.} \end{cases}$$

(ii) The pre-image of (N, E) under (φ, ψ), denoted by $(\varphi, \psi)^{-1}(N, E)$, is an NSS over U and is defined by :
$(\varphi, \psi)^{-1}(N, E) = (\varphi^{-1}(N), \psi^{-1}(E))$ where $\forall a \in \psi^{-1}(E), \forall x \in U$,

$$\begin{aligned} T_{f_{\varphi^{-1}(N)}(a)}(x) &= T_{f_N[\psi(a)]}(\varphi(x)) \\ I_{f_{\varphi^{-1}(N)}(a)}(x) &= I_{f_N[\psi(a)]}(\varphi(x)) \\ F_{f_{\varphi^{-1}(N)}(a)}(x) &= F_{f_N[\psi(a)]}(\varphi(x)) \end{aligned}$$

If ψ and φ is injective (surjective), then (φ, ψ) is injective (surjective).

2.9 Definition [26]

1. An NSS M over (R, E) is said to be constant if each $f_M(e)$ is constant for $e \in E$ i.e., $(T_{f_M(e)}(x), I_{f_M(e)}(x), F_{f_M(e)}(x))$ is same $\forall e \in E$, $\forall x \in R$.

For M to be nonconstant, if for each $e \in E$ the triplet $(T_{f_M(e)}(x), I_{f_M(e)}(x), F_{f_M(e)}(x))$ is atleast of two different kinds $\forall x \in R$.

2. Let R be a ring and M, N be two NSSs over (R, E). Then $MoN = L$ (say) is also an NSS over (R, E) and is defined as following, for $e \in E$ and $x \in R$,

$$T_{f_L(e)}(x) = \begin{cases} \max_{x=yz}[T_{f_M(e)}(y) * T_{f_N(e)}(z)] \\ 0 \quad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

$$I_{f_L(e)}(x) = \begin{cases} \min_{x=yz}[I_{f_M(e)}(y) \diamond I_{f_N(e)}(z)] \\ 1 \quad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

$$F_{f_L(e)}(x) = \begin{cases} \min_{x=yz}[F_{f_M(e)}(y) \diamond F_{f_N(e)}(z)] \\ 1 \quad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

3. A neutrosophic soft ideal P over (R, E) is said to be a neutrosophic soft prime ideal if (i) P is not constant neutrosophic soft ideal, (ii) for any two neutrosophic soft ideals M, N over (R, E), $MoN \subseteq P \Rightarrow$ either $M \subseteq P$ or $N \subseteq P$.

2.10 Theorem [26]

1. Let P be an NSS over (R, E) such that cardinality of $f_P(e)$ is 2 i.e., $|f_P(e)| = 2$ and $[f_P(e)](0_r) = (1, 0, 0)$ for each $e \in E$. If $P_0 = \{x \in R : [f_P(e)](x) = [f_P(e)](0_r)\}$ is a prime ideal over R, then P is a neutrosophic soft prime ideal over (R, E).

2. Let P be an NSS over (R, E). Then P is a neutrosophic soft left (right) ideal over (R, E) iff $\widehat{P} = \{x \in R : [f_P(e)](x) = (1, 0, 0)\}$ with $0_r \in \widehat{P}$ is a left (right) ideal of R.

3. $S(\neq \phi) \subset R$ is an ideal of R iff there exists a neutrosophic soft ideal M over (R, E) where $f_M : E \longrightarrow NS(R)$ is defined as, $\forall e \in E$,

$$[f_M(e)](x) = \begin{cases} (r_1, r_2, r_3) & \text{if } x \in S \\ (t_1, t_2, t_3) & \text{if } x \notin S. \end{cases}$$

with $r_1 > t_1$, $r_2 < t_2$, $r_3 < t_3$ and $r_1, r_2, r_3, t_1, t_2, t_3 \in [0, 1]$.
In particular, $S(\neq \phi) \subset R$ is an ideal of R iff the characteristic function χ_S is a

neutrosophic soft ideal over (R, E) where $\chi_S : E \longrightarrow NS(R)$ is defined as, $\forall e \in E$,

$$[\chi_S(e)](x) = \begin{cases} (1,0,0) & \text{if } x \in S \\ (0,1,1) & \text{if } x \notin S. \end{cases}$$

4. An NSS M over (R, E) is a neutrosophic soft left (right) ideal iff each nonempty level set $[f_M(e)]_{(\alpha,\beta,\gamma)}$ of the neutrosophic set $f_M(e)$ is a left (right) ideal of R where $\alpha \in Im\, T_{f_M(e)}$, $\beta \in Im\, I_{f_M(e)}$, $\gamma \in Im\, F_{f_M(e)}$.

5. Let P be a neutrosophic soft left (right) ideal over (R, E). Then $P_0 = \{x \in R : [f_P(e)](x) = [f_P(e)](0_r)\}$ is a left (right) ideal of R.

6. Let P be a neutrosophic soft prime ideal over (R, E). Then $P_0 = \{x \in R : [f_P(e)](x) = [f_P(e)](0_r)\}$ is a prime ideal of R.

2.11 Definition [7]

A left k-ideal I of a semiring S is a left ideal such that if $a \in I$ and $x \in S$ and if either $a + x \in I$ or $x + a \in I$, then $x \in I$.
Right k-ideal of a semiring is defined dually. A non-empty subset I of a semiring S is called a k-ideal if it is both a left k-ideal and a right k-ideal.

3 Neutrosophic soft completely prime ideal

Here first we have defined a completely prime ideal of a ring and then defined a neutrosophic soft completely prime ideal. These are illustrated with suitable examples. Along with several related properties and theorems have been developed.

Through out this paper, unless otherwise stated, E is treated as the parametric set and $e \in E$, an arbitrary parameter. Moreover the standard t-norm and s-norm are taken into consideration wherever needed through out this paper i.e., $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$.

3.1 Definition

An ideal S of a ring R is called a completely prime ideal of R if for $x, y \in R$, $xy \in S \Rightarrow$ either $x \in S$ or $y \in S$.

3.1.1 *Example*

1. For the ring $(\mathbf{Z}, +, \cdot)$ ($\mathbf{Z}$ being the set of integers), an ideal $(2\mathbf{Z}, +, \cdot)$ is a completely prime ideal.

2. We assume a ring $R = \{0, x, y, z\}$. The two binary operations addition and multiplication on R are given by the following tables :

Table 1

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 2

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	y	y
z	0	0	y	y

It is an abelian ring. With respect to these two tables, $\{0, x\}$ and $\{0, y\}$ are two ideals of R. From 2nd table, it is evident that $\{0, x\}$ is a completely prime ideal of R but $\{0, y\}$ is not so because $z \cdot z = y$ though $z \notin \{0, y\}$.

3. Consider the another ring $R = \{0, x, y, z\}$ with two binary operations addition and multiplication on R are given by the following tables :

Table 3

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 4

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	x	y	x

It is not an abelian ring. With respect to these two tables, $\{0, x\}$ is an ideal of R but not completely prime ideal. Because $y \cdot z = 0, z \cdot z = x, y \cdot y = 0$ but $y, z \notin \{0, x\}$.

3.2 Proposition

If S is a completely prime ideal of a ring R then S is a prime ideal of R.

Proof. Let S be a completely prime ideal of a ring R and A, B be two ideals of R such that $AB \subseteq S$. Suppose $A \not\subseteq S$ and $B \not\subseteq S$. Then there exists $x \in A$ and $y \in B$ such that $x, y \notin S$. But $xy \in S$ as $AB \subseteq S$. Since S is a completely prime ideal of R, so either $x \in S$ or $y \in S$ and this leads a contradiction to the fact $x, y \notin S$. Hence S is a prime ideal of R.

3.3 Definition

A neutrosophic soft ideal N over (R, E) is called a neutrosophic soft completely prime ideal if $\forall x, y \in R$ and $\forall e \in E$,

$$\begin{cases} T_{f_N(e)}(x \cdot y) \leq \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\} \\ I_{f_N(e)}(x \cdot y) \geq \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\} \\ F_{f_N(e)}(x \cdot y) \geq \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\}. \end{cases}$$

3.3.1 *Example*

Consider the Example [3.1.1](2). We define an NSS M over (R, E) as following, $\forall r \in R$ and $\forall e \in E$,

$$[f_M(e)](r) = \begin{cases} (1, 0.3, 0.1) & \text{if } r \in \{0, x\} \\ (0.8, 0.6, 0.4) & \text{if } r \notin \{0, x\}. \end{cases}$$

Then M is a neutrosophic soft completely prime ideal over (R, E).

3.4 Theorem

An NSS N is a neutrosophic soft completely prime ideal over (R, E) iff for $e \in E$, $|f_N(e)| = 2$, $[f_N(e)](0_r) = (1, 0, 0)$ and $\widehat{N} = \{x \in R : [f_N(e)](x) = (1, 0, 0)\}$ is a

completely prime ideal of R.

Proof. Let N be a neutrosophic soft completely prime ideal over (R, E). Then N is a neutrosophic soft ideal over (R, E) and so $\widehat{N}$ is an ideal over R by Theorem [2.11](2). To prove $\widehat{N}$ is a complete prime ideal, let $xy \in \widehat{N}$ for $x, y \in R$. Then $[f_N(e)](xy) = (1, 0, 0)$ for $e \in E$. But,

$$1 = T_{f_N(e)}(xy) \leq \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\},$$
$$0 = I_{f_N(e)}(xy) \geq \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\},$$
$$0 = F_{f_N(e)}(xy) \geq \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\};$$

This implies that

$$T_{f_N(e)}(0_r) = 1 \leq \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\},$$
$$I_{f_N(e)}(0_r) = 0 \geq \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\},$$
$$F_{f_N(e)}(0_r) = 0 \geq \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\};$$

This shows that,

$$\text{either } T_{f_N(e)}(0_r) \leq T_{f_N(e)}(x) \text{ or } T_{f_N(e)}(0_r) \leq T_{f_N(e)}(y),$$
$$\text{either } I_{f_N(e)}(0_r) \geq I_{f_N(e)}(x) \text{ or } I_{f_N(e)}(0_r) \geq I_{f_N(e)}(y),$$
$$\text{either } F_{f_N(e)}(0_r) \geq F_{f_N(e)}(x) \text{ or } F_{f_N(e)}(0_r) \geq F_{f_N(e)}(y);$$

But $T_{f_N(e)}(0_r) \geq T_{f_N(e)}(x)$, $I_{f_N(e)}(0_r) \leq I_{f_N(e)}(x)$, $F_{f_N(e)}(0_r) \leq F_{f_N(e)}(x)$, $\forall x \in R$. Hence $T_{f_N(e)}(x) = T_{f_N(e)}(0_r)$, $I_{f_N(e)}(x) = I_{f_N(e)}(0_r)$, $F_{f_N(e)}(x) = F_{f_N(e)}(0_r)$, $\forall x \in R$ i.e., $x, y \in \widehat{N}$. Thus $\widehat{N}$ is a complete prime ideal.

Conversely suppose $\widehat{N}$ is a completely prime ideal with the given conditions. As $\widehat{N}$ is an ideal of R, so N is a neutrosophic soft ideal over (R, E) by Theorem [2.11](2). For contrary, suppose N is not neutrosophic soft completely prime ideal. Then,

$$T_{f_N(e)}(xy) > \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\},$$
$$I_{f_N(e)}(xy) < \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\},$$
$$F_{f_N(e)}(xy) < \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\};$$

Since $|f_N(e)| = 2$ and $[f_N(e)](0_r) = (1, 0, 0)$ then there exists $x, y \in R$ so that $[f_N(e)](x) = [f_N(e)](y) = (r_1, r_2, r_3) \neq (1, 0, 0)$ (say) for $0 \leq r_1 < 1$ and $0 < r_2, r_3 \leq 1$. Then,

$$\begin{aligned} & T_{f_N(e)}(xy) > r_1,\ I_{f_N(e)}(xy) < r_2,\ F_{f_N(e)}(xy) < r_3 \\ \Rightarrow\ & T_{f_N(e)}(xy) = 1,\ I_{f_N(e)}(xy) = F_{f_N(e)}(xy) = 0 \\ \Rightarrow\ & [f_N(e)](xy) = (1, 0, 0) \\ \Rightarrow\ & xy \in \widehat{N} \end{aligned}$$

Since $\widehat{N}$ is completely prime ideal, so either $x \in \widehat{N}$ or $y \in \widehat{N}$ i.e., $[f_N(e)](x) = [f_N(e)](y) = (1, 0, 0)$. A contradiction arises to the fact that $[f_N(e)](x) = [f_N(e)](y) = (r_1, r_2, r_3) \neq (1, 0, 0)$. Thus,

$$T_{f_N(e)}(xy) \leq \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\},$$
$$I_{f_N(e)}(xy) \geq \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\},$$
$$F_{f_N(e)}(xy) \geq \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\};$$

and so N is a neutrosophic soft completely prime ideal over (R, E).

3.5 Theorem

Let N be a neutrosophic soft completely prime ideal over (R, E) with $|f_N(e)| = 2$, $[f_N(e)](0_r) = (1, 0, 0)$ for each $e \in E$. Then N is a neutrosophic soft prime ideal over (R, E).

Proof. Let the condition hold. By Theorem [3.4], $\widehat{N} = \{x \in R : [f_N(e)](x) = (1, 0, 0)\}$ is a completely prime ideal of R. Then by Proposition [3.2], $\widehat{N}$ is a prime ideal of R. Hence N is a neutrosophic soft prime ideal over (R, E) by Theorem [2.11](1).

3.6 Theorem

Let R be a ring. Then $S(\neq \phi) \subset R$ be a completely prime ideal of R iff an NSS N over (R, E) is a neutrosophic soft completely prime ideal where $f_N : E \longrightarrow NS(R)$ is defined as :

$$[f_N(e)](x) = \begin{cases} (r_1, r_2, r_3) & \text{if } x \in S \\ (t_1, t_2, t_3) & \text{if } x \notin S. \end{cases}$$

with $r_1 > t_1$, $r_2 < t_2$, $r_3 < t_3$ and $r_1, r_2, r_3, t_1, t_2, t_3 \in [0, 1]$.

Proof. First let $S(\neq \phi) \subset R$ be a completely prime ideal of R. Then S is an ideal of R and so by Theorem [2.11](3), N is a neutrosophic soft ideal over (R, E). To end the theorem, we shall just show that N is completely prime. For contrary, suppose

$$\begin{aligned} &T_{f_N(e)}(xy) > \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\}, \\ &I_{f_N(e)}(xy) < \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\}, \\ &F_{f_N(e)}(xy) < \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\}; \end{aligned}$$

Then by definition of $f_N(e)$, we have $[f_N(e)](xy) = (r_1, r_2, r_3)$ and $[f_N(e)](x) = [f_N(e)](y) = (t_1, t_2, t_3)$. This implies $xy \in S$ but $x, y \notin S$ which is a contradiction to the fact that S is a completely prime ideal of R. Hence N is a neutrosophic soft completely prime ideal over (R, E).

Conversely, let N in given form be a neutrosophic soft completely prime ideal over (R, E). Then N is a neutrosophic soft ideal over (R, E) and so by Theorem [2.11](3), S is an ideal of R. To show S is a completely prime ideal of R, let $xy \in S$. Then,

$$\begin{aligned} & [f_N(e)](xy) = (r_1, r_2, r_3) \\ \Rightarrow\ & T_{f_N(e)}(xy) = r_1,\ I_{f_N(e)}(xy) = r_2,\ F_{f_N(e)}(xy) = r_3 \\ \Rightarrow\ & \max\{T_{f_N(e)}(x), T_{f_N(e)}(y)\} \geq r_1,\ \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\} \leq r_2, \\ & \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\} \leq r_3 \\ \Rightarrow\ & \text{either } T_{f_N(e)}(x) \geq r_1,\ I_{f_N(e)}(x) \leq r_2,\ F_{f_N(e)}(x) \leq r_3 \\ & \text{or } T_{f_N(e)}(y) \geq r_1,\ I_{f_N(e)}(y) \leq r_2,\ F_{f_N(e)}(y) \leq r_3 \\ \Rightarrow\ & \text{either } x \in S \text{ or } y \in S \end{aligned}$$

Thus S is a completely prime ideal of R.

3.6.1 Corollary

A non empty subset S of a ring R is a completely prime ideal iff the characteristic function χ_S is a neutrosophic soft completely prime ideal over (R, E) where $\chi_S : E \longrightarrow NS(R)$ is defined by :

$$[\chi_S(e)](x) = \begin{cases} (1,0,0) & \text{if } x \in S \\ (0,1,1) & \text{if } x \notin S. \end{cases}$$

Proof. It is the particular case of Theorem [3.6].

3.7 Theorem

An NSS M over (R, E) is a neutrosophic soft completely prime ideal means each nonempty level set $[f_M(e)]_{(\alpha,\beta,\gamma)}$ of the neutrosophic set $f_M(e)$, $e \in E$ is a completely prime ideal of R where $\alpha \in Im\, T_{f_M(e)}$, $\beta \in Im\, I_{f_M(e)}$, $\gamma \in Im\, F_{f_M(e)}$.

Proof. Here M is a neutrosophic soft completely prime ideal over (R, E). Then M is a neutrosophic soft ideal over (R, E) and so by Theorem [2.11](4), $[f_M(e)]_{(\alpha,\beta,\gamma)}$ is an ideal of R. To complete the theorem, let $xy \in [f_M(e)]_{(\alpha,\beta,\gamma)}$. Then,

$$\begin{aligned} & T_{f_M(e)}(xy) \geq \alpha,\ I_{f_M(e)}(xy) \leq \beta,\ F_{f_M(e)}(xy) \leq \gamma \\ \Rightarrow\ & \max\{T_{f_M(e)}(x), T_{f_M(e)}(y)\} \geq \alpha,\ \min\{I_{f_M(e)}(x), I_{f_M(e)}(y)\} \leq \beta, \\ & \min\{F_{f_M(e)}(x), F_{f_M(e)}(y)\} \leq \gamma \\ \Rightarrow\ & \text{either } T_{f_M(e)}(x) \geq \alpha,\ I_{f_M(e)}(x) \leq \beta,\ F_{f_M(e)}(x) \leq \gamma \\ & \text{or } T_{f_M(e)}(y) \geq \alpha,\ I_{f_M(e)}(y) \leq \beta,\ F_{f_M(e)}(y) \leq \gamma \\ \Rightarrow\ & \text{either } x \in [f_M(e)]_{(\alpha,\beta,\gamma)} \text{ or } y \in [f_M(e)]_{(\alpha,\beta,\gamma)} \end{aligned}$$

Thus $[f_M(e)]_{(\alpha,\beta,\gamma)}$ is a completely prime ideal of R.

3.8 Proposition

Let S be a completely prime ideal of a ring R. Then there exists a neutrosophic soft completely prime ideal M over (R, E) such that $[f_M(e)]_{(\alpha,\beta,\gamma)} = S$ for $e \in E$ and $\alpha, \beta, \gamma \in (0, 1)$.

Proof. As S is a completely prime ideal of a ring R, so S is an ideal of R. For $\alpha, \beta, \gamma \in (0, 1)$ define an NSS M over (R, E) as following :

$$[f_M(e)](x) = \begin{cases} (\alpha,\beta,\gamma) & \text{if } x \in S \\ (0,1,1) & \text{if } x \notin S. \end{cases}$$

Then by Theorem [2.11](3), M is a neutrosophic soft ideal over (R, E). If possible let M is not a neutrosophic soft completely prime ideal over (R, E). Then,

$$\begin{aligned} & T_{f_M(e)}(xy) > \max\{T_{f_M(e)}(x), T_{f_M(e)}(y)\}, \\ & I_{f_M(e)}(xy) < \min\{I_{f_M(e)}(x), I_{f_M(e)}(y)\}, \\ & F_{f_M(e)}(xy) < \min\{F_{f_M(e)}(x), F_{f_M(e)}(y)\}; \end{aligned}$$

Then by definition of $f_M(e)$, we have $[f_M(e)](xy) = (\alpha, \beta, \gamma)$ and $[f_M(e)](x) = [f_M(e)](y) = (0, 1, 1)$. This implies $xy \in S$ but $x, y \notin S$ which is a contradiction to the fact that S is a completely prime ideal of R. Hence M is a neutrosophic soft completely prime ideal over (R, E). Obviously $[f_M(e)]_{(\alpha,\beta,\gamma)} = S$ for each $e \in E$.

3.9 Theorem

Let $(\varphi, \psi) : (R_1, E) \longrightarrow (R_2, E)$ be a neutrosophic soft homomorphism where R_1, R_2 be two rings. Suppose (M, E) and (N, E) be two neutrosophic soft left (right) ideals over R_1 and R_2, respectively. Then,
1. $(\varphi, \psi)(M, E)$ is a neutrosophic soft left (right) ideal over R_2 if (φ, ψ) is epimorphism.
2. $(\varphi, \psi)^{-1}(N, E)$ is a neutrosophic soft left (right) ideal over R_1.

Proof. **1.** Let $b \in \psi(E)$ and $y_1, y_2, s \in R_2$. For $\varphi^{-1}(y_1) = \phi$ or $\varphi^{-1}(y_2) = \phi$, the proof is straight forward.
So, we assume that there exists $x_1, x_2, r \in R_1$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(r) = s$. Then,

$$\begin{aligned}
T_{f_{\varphi(M)}(b)}(y_1 - y_2) &= \max_{\varphi(x)=y_1-y_2} \max_{\psi(a)=b} [T_{f_M(a)}(x)] \\
&\geq \max_{\psi(a)=b} [T_{f_M(a)}(x_1 - x_2)] \\
&\geq \max_{\psi(a)=b} [T_{f_M(a)}(x_1) * T_{f_M(a)}(x_2)] \\
&= \max_{\psi(a)=b} [T_{f_M(a)}(x_1)] * \max_{\psi(a)=b} [T_{f_M(a)}(x_2)] \\
T_{f_{\varphi(M)}(b)}(sy_1) &= \max_{\varphi(x)=sy_1} \max_{\psi(a)=b} [T_{f_M(a)}(x)] \\
&\geq \max_{\psi(a)=b} [T_{f_M(a)}(rx_1)] \\
&\geq \max_{\psi(a)=b} [T_{f_M(a)}(x_1)]
\end{aligned}$$

Since, this inequality is satisfied for each $x_1, x_2 \in R_1$ satisfying $\varphi(x_1) = y_1, \varphi(x_2) = y_2$ so we have,

$$\begin{aligned}
& T_{f_{\varphi(M)}(b)}(y_1 - y_2) \\
\geq\ & (\max_{\varphi(x_1)=y_1} \max_{\psi(a)=b} [T_{f_M(a)}(x_1)]) * (\max_{\varphi(x_2)=y_2} \max_{\psi(a)=b} [T_{f_M(a)}(x_2)]) \\
=\ & T_{f_{\varphi(M)}(b)}(y_1) * T_{f_{\varphi(M)}(b)}(y_2)
\end{aligned}$$

Also, $T_{f_{\varphi(M)}(b)}(sy_1) \geq \max_{\varphi(x_1)=y_1} \max_{\psi(a)=b} [T_{f_M(a)}(x_1)] = T_{f_{\varphi(M)}(b)}(y_1)$

Next,

$$\begin{aligned}
I_{f_{\varphi(M)}(b)}(y_1 - y_2) &= \min_{\varphi(x)=y_1-y_2} \min_{\psi(a)=b} [I_{f_M(a)}(x)] \\
&\leq \min_{\psi(a)=b} [I_{f_M(a)}(x_1 - x_2)] \\
&\leq \min_{\psi(a)=b} [I_{f_M(a)}(x_1) \diamond I_{f_M(a)}(x_2)] \\
&= \min_{\psi(a)=b} [I_{f_M(a)}(x_1)] \diamond \min_{\psi(a)=b} [I_{f_M(a)}(x_2)]
\end{aligned}$$

$$\begin{aligned} I_{f_{\varphi(M)(b)}}(sy_1) &= \min_{\varphi(x)=sy_1} \min_{\psi(a)=b} [I_{f_M(a)}(x)] \\ &\leq \min_{\psi(a)=b} [I_{f_M(a)}(rx_1)] \\ &\leq \min_{\psi(a)=b} [I_{f_M(a)}(x_1)] \end{aligned}$$

Since, this inequality is satisfied for each $x_1, x_2 \in R_1$ satisfying $\varphi(x_1) = y_1, \varphi(x_2) = y_2$ so we have,

$$\begin{aligned} & I_{f_{\varphi(M)(b)}}(y_1 - y_2) \\ \leq\ & (\min_{\varphi(x_1)=y_1} \min_{\psi(a)=b} [I_{f_M(a)}(x_1)]) \diamond (\min_{\varphi(x_2)=y_2} \min_{\psi(a)=b} [I_{f_M(a)}(x_2)]) \\ =\ & I_{f_{\varphi(M)(b)}}(y_1) \diamond I_{f_{\varphi(M)(b)}}(y_2) \end{aligned}$$

Also, $I_{f_{\varphi(M)(b)}}(sy_1) \leq \min_{\varphi(x_1)=y_1} \min_{\psi(a)=b} [I_{f_M(a)}(x_1)] = I_{f_{\varphi(M)(b)}}(y_1)$.
Similarly, we can show that
$F_{f_{\varphi(M)(b)}}(y_1 - y_2) \leq F_{f_{\varphi(M)(b)}}(y_1) \diamond F_{f_{\varphi(M)(b)}}(y_2)$, $F_{f_{\varphi(M)(b)}}(sy_1) \geq F_{f_{\varphi(M)(b)}}(y_1)$;
This completes the proof.

2. For $a \in \psi^{-1}(E)$ and $x_1, x_2 \in R_1$, we have,

$$\begin{aligned} T_{f_{\varphi^{-1}(N)(a)}}(x_1 - x_2) &= T_{f_N[\psi(a)]}(\varphi(x_1 - x_2)) \\ &= T_{f_N[\psi(a)]}(\varphi(x_1) - \varphi(x_2)) \\ &\geq T_{f_N[\psi(a)]}(\varphi(x_1)) * T_{f_N[\psi(a)]}(\varphi(x_2)) \\ &= T_{f_{\varphi^{-1}(N)(a)}}(x_1) * T_{f_{\varphi^{-1}(N)(a)}}(x_2) \\ T_{f_{\varphi^{-1}(N)(a)}}(rx_1) &= T_{f_N[\psi(a)]}(\varphi(rx_1)) \\ &= T_{f_N[\psi(a)]}(\varphi(r)\varphi(x_1)) \\ &\geq T_{f_N[\psi(a)]}(s\varphi(x_1)) \\ &\geq T_{f_N[\psi(a)]}(\varphi(x_1)) \\ &= T_{f_{\varphi^{-1}(N)(a)}}(x_1) \end{aligned}$$

Next,

$$\begin{aligned} I_{f_{\varphi^{-1}(N)(a)}}(x_1 - x_2) &= I_{f_N[\psi(a)]}(\varphi(x_1 - x_2)) \\ &= I_{f_N[\psi(a)]}(\varphi(x_1) - \varphi(x_2)) \\ &\leq I_{f_N[\psi(a)]}(\varphi(x_1)) \diamond I_{f_N[\psi(a)]}(\varphi(x_2)) \\ &= I_{f_{\varphi^{-1}(N)(a)}}(x_1) \diamond I_{f_{\varphi^{-1}(N)(a)}}(x_2) \\ I_{f_{\varphi^{-1}(N)(a)}}(rx_1) &= I_{f_N[\psi(a)]}(\varphi(rx_1)) \\ &= I_{f_N[\psi(a)]}(\varphi(r)\varphi(x_1)) \\ &\leq I_{f_N[\psi(a)]}(s\varphi(x_1)) \\ &\leq I_{f_N[\psi(a)]}(\varphi(x_1)) \\ &= I_{f_{\varphi^{-1}(N)(a)}}(x_1) \end{aligned}$$

Similarly, $F_{f_{\varphi^{-1}(N)(a)}}(x_1 - x_2) \leq F_{f_{\varphi^{-1}(N)(a)}}(x_1) \diamond F_{f_{\varphi^{-1}(N)(a)}}(x_2)$ and
$F_{f_{\varphi^{-1}(N)(a)}}(rx_1) \leq F_{f_{\varphi^{-1}(N)(a)}}(x_1)$;
This proves the 2nd part.

3.10 Theorem

Let (φ, ψ) be a neutrosophic soft homomorphism from a ring R_1 to a ring R_2. Suppose (M, E) and (N, E) are neutrosophic soft completely prime ideals over R_1 and R_2, respectively. Then,
1. $(\varphi, \psi)(M, E)$ is a neutrosophic soft completely prime ideal over R_2.
2. $(\varphi, \psi)^{-1}(N, E)$ is a neutrosophic soft completely prime ideal over R_1.

Proof. **1.** If possible, let (M, E) be a neutrosophic soft completely prime ideal over R_1 but $(\varphi, \psi)(M, E)$ is not so over R_2. Then for $b \in \psi(E)$ and $y_1, y_2 \in R_2$,

$$\begin{aligned}
& T_{f_{\varphi(M)(b)}}(y_1y_2) > \max\{T_{f_{\varphi(M)(b)}}(y_1), T_{f_{\varphi(M)(b)}}(y_2)\} \\
\Rightarrow \quad & \max_{\varphi(x)=y_1y_2} \max_{\psi(a)=b} [T_{f_M(a)}(x)] > \max\{(\max_{\varphi(x)=y_1} \max_{\psi(a)=b} [T_{f_M(a)}(x)]), \\
& (\max_{\varphi(x)=y_2} \max_{\psi(a)=b} [T_{f_M(a)}(x)])\} \\
\Rightarrow \quad & \max_{\varphi(x)=y_1y_2} [T_{f_M(a)}(x)] > \max\{(\max_{\varphi(x)=y_1} [T_{f_M(a)}(x)]), (\max_{\varphi(x)=y_2} [T_{f_M(a)}(x)])\} \\
\Rightarrow \quad & \max_{\varphi(x)=y_1y_2} [T_{f_M(a)}(x)] \geq \max\{T_{f_M(a)}(x_1), T_{f_M(a)}(x_2)\}
\end{aligned}$$

Since the inequality holds for each $x_1, x_2 \in R_1$ satisfying $\varphi(x_1) = y_1, \varphi(x_2) = y_2$ so we have $T_{f_M(a)}(x_1x_2) > \max\{T_{f_M(a)}(x_1), T_{f_M(a)}(x_2)\}$ which is a contradiction to the truth that (M, E) is a neutrosophic soft completely prime ideal over R_1. We can reach to the same conclusion taking the indeterminacy membership function (I) and falsity membership function (F) also. Hence we get the first result.

2. For $a \in \psi^{-1}(E)$ and $x_1, x_2 \in R_1$, we have,

$$\begin{aligned}
T_{f_{\varphi^{-1}(N)}(a)}(x_1x_2) &= T_{f_N[\psi(a)]}(\varphi(x_1x_2)) \\
&= T_{f_N[\psi(a)]}(\varphi(x_1)\varphi(x_2)) \\
&\leq \max\{T_{f_N[\psi(a)]}(\varphi(x_1)), T_{f_N[\psi(a)]}(\varphi(x_2))\} \\
&= \max\{T_{f_{\varphi^{-1}(N)}(a)}(x_1), T_{f_{\varphi^{-1}(N)}(a)}(x_2)\} \\
I_{f_{\varphi^{-1}(N)}(a)}(x_1x_2) &= I_{f_N[\psi(a)]}(\varphi(x_1x_2)) \\
&= I_{f_N[\psi(a)]}(\varphi(x_1)\varphi(x_2)) \\
&\geq \min\{I_{f_N[\psi(a)]}(\varphi(x_1)), I_{f_N[\psi(a)]}(\varphi(x_2))\} \\
&= \min\{I_{f_{\varphi^{-1}(N)}(a)}(x_1), I_{f_{\varphi^{-1}(N)}(a)}(x_2)\} \\
F_{f_{\varphi^{-1}(N)}(a)}(x_1x_2) &= F_{f_N[\psi(a)]}(\varphi(x_1x_2)) \\
&= F_{f_N[\psi(a)]}(\varphi(x_1)\varphi(x_2)) \\
&\geq \min\{F_{f_N[\psi(a)]}(\varphi(x_1)), F_{f_N[\psi(a)]}(\varphi(x_2))\} \\
&= \min\{F_{f_{\varphi^{-1}(N)}(a)}(x_1), F_{f_{\varphi^{-1}(N)}(a)}(x_2)\}
\end{aligned}$$

This shows the 2nd result.

4 Neutrosophic Soft Completely Semi-Prime Ideal

In this section the concept of semi-prime ideal, completely semi-prime ideal of a ring R and neutrosophic soft completely semi-prime ideal are focussed.

4.1 Definition

1. An ideal I of a ring R is called a semi-prime ideal if there is another ideal J of R such that $JJ \subseteq I \Rightarrow J \subseteq I$.

2. An ideal J of a ring R is called a completely semi-prime ideal if for $x \in R$, $xx \in J \Rightarrow x \in J$. xx is denoted by x^2.

4.1.1 *Example*

1. Let $R = \{0, x, y, z\}$ be a ring. The two binary operations addition and multiplication on R are given by the following tables :

Table 5

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 6

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	x	0
y	0	x	y	z
z	0	0	z	z

Then $\{0, x\}$ is a completely semi-prime ideal of R as $0 \cdot 0 = 0, x \cdot x = x, y \cdot y = y, z \cdot z = z$.

2. Consider the Example [3.1.1](3). Then $\{0, x\}$ is not a completely semi-prime ideal, because $z \cdot z = x, y \cdot y = 0$ but $y, z \notin \{0, x\}$.

4.2 Proposition

Every completely prime ideal of a ring R is a completely semi-prime ideal of R.

Proof. By taking $y = x$, the proof follows directly from Definition [3.1].

4.3 Definition

Let R be a ring and E be a parametric set. A neutrosophic soft ideal N over (R, E) is called a neutrosophic soft completely semi-prime ideal if $\forall x, y \in R$ and $\forall e \in E$, $T_{f_N(e)}(x^2) \leq T_{f_N(e)}(x)$, $I_{f_N(e)}(x^2) \geq I_{f_N(e)}(x)$, $F_{f_N(e)}(x^2) \geq F_{f_N(e)}(x)$.

4.3.1 *Example*

Consider the Example [4.1.1](1). We define an NSS M over (R, E) as following, $\forall r \in R$ and $\forall e \in E$,

$$[f_M(e)](r) = \begin{cases} (0.4, 0.1, 0.5) & \text{if } r \in \{0, x\} \\ (0.2, 0.5, 0.8) & \text{if } r \notin \{0, x\}. \end{cases}$$

Then M is a neutrosophic soft completely semi-prime ideal over (R, E).

4.4 *Lemma*

A neutrosophic soft ideal N over (R, E) is a neutrosophic soft completely semi-prime ideal iff $[f_N(e)](x^2) = [f_N(e)](x)$, for every $e \in E, x \in R$.

Proof. Let N be a neutrosophic soft ideal over (R, E) with $[f_N(e)](x^2) = [f_N(e)](x)$, $\forall e \in E$ and $\forall x \in R$. Then by Definition [4.3], N is a neutrosophic soft completely semi-prime ideal over (R, E).

Conversely, if N is a neutrosophic soft completely semi-prime ideal by Definition [4.3], $T_{f_N(e)}(x^2) \leq T_{f_N(e)}(x)$, $I_{f_N(e)}(x^2) \geq I_{f_N(e)}(x)$, $F_{f_N(e)}(x^2) \geq F_{f_N(e)}(x)$ and as N is a neutrosophic soft ideal over (R, E), then $T_{f_N(e)}(x^2) \geq T_{f_N(e)}(x)$, $I_{f_N(e)}(x^2) \leq I_{f_N(e)}(x)$, $F_{f_N(e)}(x^2) \leq F_{f_N(e)}(x)$. Hence $[f_N(e)](x^2) = [f_N(e)](x)$ for every $e \in E, x \in R$.

4.5 Theorem

An NSS N over (R, E) is a neutrosophic soft completely semi-prime ideal iff for $e \in E$, $S = \{x \in R : [f_N(e)](x) = [f_N(e)](0_r)\}, 0_r$ being the additive identity of ring R, is a completely semi-prime ideal of R.

Proof. Let N be a neutrosophic soft completely semi-prime ideal over (R, E). Then $[f_N(e)](x^2) = [f_N(e)](x)$ for every $e \in E, x \in R$. Now let $x^2 \in S$. Then $[f_N(e)](x^2) = [f_N(e)](0_r) \Rightarrow [f_N(e)](x) = [f_N(e)](0_r) \Rightarrow x \in S$. Hence S is a completely semi-prime ideal of R.

Conversely, if S is a completely semi-prime ideal of R. Then $x^2 \in S \Rightarrow x \in S$. Since $x^2 \in S$, then $[f_N(e)](x^2) = [f_N(e)](0_r)$ and $[f_N(e)](x) = [f_N(e)](0_r) \Rightarrow [f_N(e)](x^2) = [f_N(e)](x)$. Hence by Lemma [4.4], N is a neutrosophic soft completely semi-prime ideal over (R, E).

4.6 Theorem

An NSS N is a neutrosophic soft completely semi-prime ideal over (R, E) iff $[f_N(e)]_{(\alpha,\beta,\gamma)}$ is a completely semi-prime ideal of R where $\alpha \in Im\, T_{f_N(e)}$, $\beta \in Im\, I_{f_N(e)}$, $\gamma \in Im\, F_{f_N(e)}$.

Proof. Let N be a neutrosophic soft completely semi-prime ideal over (R, E). Then $[f_N(e)](x^2) = [f_N(e)](x)$. Now,

$$\begin{aligned} & x^2 \in [f_N(e)]_{(\alpha,\beta,\gamma)} \\ \Rightarrow\ & T_{f_N(e)}(x^2) \geq \alpha, I_{f_N(e)}(x^2) \leq \beta, F_{f_N(e)}(x^2) \leq \gamma \\ \Rightarrow\ & T_{f_N(e)}(x) \geq \alpha, I_{f_N(e)}(x) \leq \beta, F_{f_N(e)}(x) \leq \gamma \\ \Rightarrow\ & x \in [f_N(e)]_{(\alpha,\beta,\gamma)} \end{aligned}$$

Hence, $[f_N(e)]_{(\alpha,\beta,\gamma)}$ is a completely semi-prime ideal of R.

Conversely, let $[f_N(e)]_{(\alpha,\beta,\gamma)}$ be a completely semi-prime ideal of R. Then $x^2 \in [f_N(e)]_{(\alpha,\beta,\gamma)} \Rightarrow x \in ([f_N(e)]_{(\alpha,\beta,\gamma)}$ i.e.,

$$\begin{aligned} & T_{f_N(e)}(x^2) \geq \alpha, I_{f_N(e)}(x^2) \leq \beta, F_{f_N(e)}(x^2) \leq \gamma \\ \Rightarrow\ & T_{f_N(e)}(x) \geq \alpha, I_{f_N(e)}(x) \leq \beta, F_{f_N(e)}(x) \leq \gamma \end{aligned}$$

Now, suppose $[f_N(e)](x^2) \neq [f_N(e)](x)$. Let $[f_N(e)](x) = (t_1, t_2, t_3)$. Then $x^2 \notin [f_N(e)]_{(t_1,t_2,t_3)}$ but $x \in [f_N(e)]_{(t_1,t_2,t_3)}$ which is a contradiction as $[f_N(e)]_{(\alpha,\beta,\gamma)}$ is a completely semi-prime ideal of R. Hence $[f_N(e)](x^2) = [f_N(e)](x)$ and so N is a neutrosophic soft completely semi-prime ideal over (R, E) by Lemma [4.4].

4.7 Theorem

Let (φ, ψ) be a neutrosophic soft homomorphism from a ring R_1 to a ring R_2. Suppose (M, E) and (N, E) are neutrosophic soft completely semi-prime ideals over R_1 and R_2, respectively. Then,
1. $(\varphi, \psi)(M, E)$ is a neutrosophic soft completely semi-prime ideal over R_2.
2. $(\varphi, \psi)^{-1}(N, E)$ is a neutrosophic soft completely semi-prime ideal over R_1.

Proof. **1.** If possible, let (M, E) be a neutrosophic soft completely semi-prime ideal over R_1 but $(\varphi, \psi)(M, E)$ is not so over R_2. Then for $b \in \psi(E)$ and $y \in R_2$,

$$\begin{aligned} & T_{f_{\varphi(M)}(b)}(y^2) > T_{f_{\varphi(M)}(b)}(y) \\ \Rightarrow \quad & \max_{\varphi(x)=y^2} \max_{\psi(a)=b} [T_{f_M(a)}(x)] > \max_{\varphi(x)=y} \max_{\psi(a)=b} [T_{f_M(a)}(x)] \\ \Rightarrow \quad & \max_{\varphi(x)=y^2} [T_{f_M(a)}(x)] > \max_{\varphi(x)=y} [T_{f_M(a)}(x)] \\ \Rightarrow \quad & \max_{\varphi(x)=y^2} [T_{f_M(a)}(x)] \geq T_{f_M(a)}(x) \end{aligned}$$

Since the inequality holds for each $x \in R_1$ satisfying $\varphi(x) = y$, so we have $T_{f_M(a)}(x^2) > T_{f_M(a)}(x)$ which is a contradiction to the fact that (M, E) is a neutrosophic soft completely semi-prime ideal over R_1. We can reach to the same conclusion taking the indeterminacy membership function (I) and falsity membership function (F) also. Hence we get the first result.
2. For $a \in \psi^{-1}(E)$ and $x \in R_1$, we have,

$$\begin{aligned} T_{f_{\varphi^{-1}(N)}(a)}(x^2) &= T_{f_N[\psi(a)]}(\varphi(x^2)) = T_{f_N[\psi(a)]}(\varphi(x))^2 \leq T_{f_N[\psi(a)]}(\varphi(x)) = T_{f_{\varphi^{-1}(N)}(a)}(x), \\ I_{f_{\varphi^{-1}(N)}(a)}(x^2) &= I_{f_N[\psi(a)]}(\varphi(x^2)) = I_{f_N[\psi(a)]}(\varphi(x))^2 \geq I_{f_N[\psi(a)]}(\varphi(x)) = I_{f_{\varphi^{-1}(N)}(a)}(x), \\ F_{f_{\varphi^{-1}(N)}(a)}(x^2) &= F_{f_N[\psi(a)]}(\varphi(x^2)) = F_{f_N[\psi(a)]}(\varphi(x))^2 \geq F_{f_N[\psi(a)]}(\varphi(x)) = F_{f_{\varphi^{-1}(N)}(a)}(x); \end{aligned}$$

This proves the 2nd result.

5 Neutrosophic soft prime k-ideal

5.1 Definition

A neutrosophic soft ideal N over (R, E) is said to be a neutrosophic soft k-ideal over (R, E) if $\forall x, y \in R$ and $\forall e \in E$,

$$\begin{cases} T_{f_N(e)}(x) \geq \min\{T_{f_N(e)}(x+y), T_{f_N(e)}(y)\} \\ I_{f_N(e)}(x) \leq \max\{I_{f_N(e)}(x+y), I_{f_N(e)}(y)\} \\ F_{f_N(e)}(x) \leq \max\{F_{f_N(e)}(x+y), F_{f_N(e)}(y)\}. \end{cases}$$

5.1.1 *Example*

1. Let $\mathbf{Z}$ be the set of all integers and $E = \{e_1, e_2, e_3\}$ be a parametric set. We consider an NSS N over $(\mathbf{Z}, E)$ given by the following table :

Table 7

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
$\mathbf{Z}_1$	$(0.3, 0.8, 0.5)$	$(0.4, 0.5, 0.7)$	$(0.7, 0.6, 0.4)$
$\mathbf{Z}_2$	$(0.4, 0.6, 0.3)$	$(0.6, 0.2, 0.4)$	$(0.7, 0.4, 0.2)$
$\mathbf{Z}_3$	$(0.6, 0.2, 0.1)$	$(1, 0, 0)$	$(0.9, 0.1, 0.1)$

where $\mathbf{Z}_1 = \{\pm 1, \pm 3, \pm 5, \cdots\}, \mathbf{Z}_2 = \{\pm 2, \pm 4, \pm 6, \cdots\}, \mathbf{Z}_3 = \{0\}$. Then N is a neutrosophic soft k-ideal over $(\mathbf{Z}, E)$. To verify it, we shall show
(i) $f_N(e)$ is neutrosophic subgroup of $(\mathbf{Z}, +)$ for each $e \in E$.
(ii) $f_N(e)$ is both neutrosophic left and right ideal of $\mathbf{Z}$ for each $e \in E$.
(iii) $f_N(e)$ is neutrosophic k-ideal of $\mathbf{Z}$ for each $e \in E$.

If $x \in \mathbf{Z}_1, y \in \mathbf{Z}_2$ then $x - y \in \mathbf{Z}_1$. We then write $\mathbf{Z}_1 - \mathbf{Z}_2 = \mathbf{Z}_1$ and so on.
Here $\mathbf{Z}_1 - \mathbf{Z}_1 = \mathbf{Z}_2$ or $\mathbf{Z}_3$, $\mathbf{Z}_1 - \mathbf{Z}_2 = \mathbf{Z}_1$, $\mathbf{Z}_1 - \mathbf{Z}_3 = \mathbf{Z}_3$, $\mathbf{Z}_2 - \mathbf{Z}_2 = \mathbf{Z}_2$ or $\mathbf{Z}_3$, $\mathbf{Z}_2 - \mathbf{Z}_3 = \mathbf{Z}_2$, $\mathbf{Z}_3 - \mathbf{Z}_3 = \mathbf{Z}_3$. Then Table 7 shows the result (i) obviously.

Next $\mathbf{Z}_1.\mathbf{Z}_1 = \mathbf{Z}_1$, $\mathbf{Z}_2.\mathbf{Z}_2 = \mathbf{Z}_2$, $\mathbf{Z}_3.\mathbf{Z}_3 = \mathbf{Z}_3$, $\mathbf{Z}_2.\mathbf{Z}_1 = \mathbf{Z}_1.\mathbf{Z}_2 = \mathbf{Z}_2$, $\mathbf{Z}_1.\mathbf{Z}_3 = \mathbf{Z}_3.\mathbf{Z}_1 = \mathbf{Z}_3$, $\mathbf{Z}_2.\mathbf{Z}_3 = \mathbf{Z}_3.\mathbf{Z}_2 = \mathbf{Z}_3$. Then the result (ii) also holds by Table 7.

Finally $\mathbf{Z}_1 + \mathbf{Z}_1 = \mathbf{Z}_2$ or $\mathbf{Z}_3$, $\mathbf{Z}_1 + \mathbf{Z}_2 = \mathbf{Z}_1$, $\mathbf{Z}_1 + \mathbf{Z}_3 = \mathbf{Z}_3$, $\mathbf{Z}_2 + \mathbf{Z}_2 = \mathbf{Z}_2$ or $\mathbf{Z}_3$, $\mathbf{Z}_2 + \mathbf{Z}_3 = \mathbf{Z}_2$, $\mathbf{Z}_3 + \mathbf{Z}_3 = \mathbf{Z}_3$. The Table 7 then meets the result (iii) clearly.

2. Let $\mathbf{R}$ be the set of real numbers and $E = \{e_1, e_2, e_3\}$ be a parametric set. Consider an NSS M over $(\mathbf{R}, E)$ given by the following table :

Table 8

	$f_M(e_1)$	$f_M(e_2)$	$f_M(e_3)$
$\mathbf{Q}$	$(0.6, 0.1, 0.3)$	$(0.8, 0.2, 0.4)$	$(0.5, 0.6, 0.7)$
$\mathbf{Q}^c$	$(0.5, 0.4, 0.7)$	$(0.4, 0.5, 0.6)$	$(0.3, 0.7, 1)$

where $\mathbf{Q}$ and $\mathbf{Q}^c$ are the set of rational and irrational numbers, respectively. If $x \in \mathbf{Q}, y \in \mathbf{Q}^c$ then $x - y \in \mathbf{Q}^c$. We write $\mathbf{Q} - \mathbf{Q}^c = \mathbf{Q}^c$ and so on.
Then $\mathbf{Q} - \mathbf{Q} = \mathbf{Q}$, $\mathbf{Q} - \mathbf{Q}^c = \mathbf{Q}^c$, $\mathbf{Q}^c - \mathbf{Q}^c = \mathbf{Q}$ or $\mathbf{Q}^c$. Clearly $f_M(e)$ is neutrosophic subgroup of $(\mathbf{R}, +)$ for each $e \in E$ by Table 8.
Next, $\mathbf{Q}.\mathbf{Q} = \mathbf{Q}$, $\mathbf{Q}.\mathbf{Q}^c = \mathbf{Q}^c$, $\mathbf{Q}^c.\mathbf{Q}^c = \mathbf{Q}$ or $\mathbf{Q}^c$. Then Table 8 shows that $f_M(e)$ is neutrosophic ideal of $\mathbf{R}$ for each $e \in E$.
Finally $\mathbf{Q} + \mathbf{Q} = \mathbf{Q}$, $\mathbf{Q} + \mathbf{Q}^c = \mathbf{Q}^c$, $\mathbf{Q}^c + \mathbf{Q}^c = \mathbf{Q}$ or $\mathbf{Q}^c$. Then $f_M(e)$ is neutrosophic k-ideal of $\mathbf{R}$ for each $e \in E$ by Table 8.
Hence M is a neutrosophic soft k-ideal over $(\mathbf{R}, E)$.

5.2 Definition

A neutrosophic soft k-ideal P over (R, E) is said to be a neutrosophic soft prime k-ideal if (i) P is not constant over (R, E), (ii) for any two neutrosophic soft ideals M, N over (R, E), $MoN \subseteq P \Rightarrow$ either $M \subseteq P$ or $N \subseteq P$.

5.3 Theorem

Let P be a neutrosophic soft prime k-ideal over (R, E). Then $P_0 = \{x \in R : [f_P(e)](x) = [f_P(e)](0_r), \forall e \in E\}$ is a prime k-ideal of R.

Proof. Let $x, x+y \in P_0$ for $x, y \in R$. Then $[f_P(e)](x) = [f_P(e)](x+y) = [f_P(e)](0_r)$. Since P is a neutrosophic soft k-ideal over (R, E), so $\forall e \in E$,

$$\begin{aligned}
&T_{f_P(e)}(y) \geq \min\{T_{f_P(e)}(x+y), T_{f_P(e)}(x)\} = T_{f_P(e)}(0_r),\\
&I_{f_P(e)}(y) \leq \max\{I_{f_P(e)}(x+y), I_{f_P(e)}(x)\} = I_{f_P(e)}(0_r),\\
&F_{f_P(e)}(y) \leq \max\{F_{f_P(e)}(x+y), F_{f_P(e)}(x)\} = F_{f_P(e)}(0_r);
\end{aligned}$$

But $T_{f_P(e)}(0_r) \geq T_{f_P(e)}(y)$, $I_{f_P(e)}(0_r) \leq I_{f_P(e)}(y)$, $F_{f_P(e)}(0_r) \leq F_{f_P(e)}(y)$, $\forall e \in E$. Thus $T_{f_P(e)}(y) = T_{f_P(e)}(0_r)$, $I_{f_P(e)}(y) = I_{f_P(e)}(0_r)$, $F_{f_P(e)}(y) \leq F_{f_P(e)}(0_r)$, $\forall e \in E$ i.e., $[f_P(e)](y) = [f_P(e)](0_r)$ and so $y \in P_0$. Hence P_0 is a k-ideal of R. Also by Theorem [2.11](6), P_0 is a prime ideal of R. This completes the proof.

5.4 Theorem

Let P be a neutrosophic soft prime k-ideal over $(\mathbf{Z}, E)$, $\mathbf{Z}$ being the set of integers with $P_0 = \{x \in R : [f_P(e)](x) = [f_P(e)](0), \forall e \in E\} = n\mathbf{Z}$, n being a natural number. Then $|f_P(e)| \leq r$, where r is the number of distinct positive divisor of n.

Proof. Let $a(\neq 0)$ be an integer and $d = gcd(a, n)$. Then there exists $r, s \in \mathbf{Z} - \{0\}$ such that $ns = ar + d$ or $ar = ns + d$. We shall now estimate following two cases :
Case 1 : When $ns = ar + d$, then $\forall e \in E$ and as $n \in P_0 = n\mathbf{Z}$,

$$\begin{aligned}
&T_{f_P(e)}(ar+d) = T_{f_P(e)}(ns) \geq T_{f_P(e)}(n) = T_{f_P(e)}(0) \geq T_{f_P(e)}(ar),\\
&I_{f_P(e)}(ar+d) = I_{f_P(e)}(ns) \leq I_{f_P(e)}(n) = I_{f_P(e)}(0) \leq I_{f_P(e)}(ar),\\
&F_{f_P(e)}(ar+d) = F_{f_P(e)}(ns) \leq F_{f_P(e)}(n) = F_{f_P(e)}(0) \leq F_{f_P(e)}(ar);
\end{aligned}$$

Again P is a neutrosophic soft k-ideal over $(\mathbf{Z}, E)$. So,

$$\begin{aligned}
&T_{f_P(e)}(d) \geq \min\{T_{f_P(e)}(ar+d), T_{f_P(e)}(ar)\} = T_{f_P(e)}(ar) \geq T_{f_P(e)}(a),\\
&I_{f_P(e)}(d) \leq \max\{I_{f_P(e)}(ar+d), I_{f_P(e)}(ar)\} = I_{f_P(e)}(ar) \leq I_{f_P(e)}(a),\\
&F_{f_P(e)}(d) \leq \max\{F_{f_P(e)}(ar+d), F_{f_P(e)}(ar)\} = F_{f_P(e)}(ar) \leq F_{f_P(e)}(a);
\end{aligned}$$

Case 2 : When $ar = ns + d$, then $\forall e \in E$ and as $n \in P_0 = n\mathbf{Z}$,

$$\begin{aligned}
&T_{f_P(e)}(ns+d) = T_{f_P(e)}(ar) \geq T_{f_P(e)}(a),\\
&I_{f_P(e)}(ns+d) = I_{f_P(e)}(ar) \leq I_{f_P(e)}(a),\\
&F_{f_P(e)}(ns+d) = F_{f_P(e)}(ar) \leq F_{f_P(e)}(a);
\end{aligned}$$

Again,

$$\begin{aligned}
&T_{f_P(e)}(ns) \geq T_{f_P(e)}(n) = T_{f_P(e)}(0) \geq T_{f_P(e)}(a),\\
&I_{f_P(e)}(ns) \leq I_{f_P(e)}(n) = I_{f_P(e)}(0) \leq I_{f_P(e)}(a),\\
&F_{f_P(e)}(ns) \leq F_{f_P(e)}(n) = F_{f_P(e)}(0) \leq F_{f_P(e)}(a);
\end{aligned}$$

Now as P is a neutrosophic soft k-ideal over $(\mathbf{Z}, E)$ so,

$$T_{f_P(e)}(d) \geq \min\{T_{f_P(e)}(ns+d), T_{f_P(e)}(ns)\} \geq T_{f_P(e)}(a),$$
$$I_{f_P(e)}(d) \leq \max\{I_{f_P(e)}(ns+d), I_{f_P(e)}(ns)\} \leq I_{f_P(e)}(a),$$
$$F_{f_P(e)}(d) \leq \max\{F_{f_P(e)}(ns+d), F_{f_P(e)}(ns)\} \leq F_{f_P(e)}(a);$$

Thus in either case $\forall e \in E$,
$T_{f_P(e)}(d) \geq T_{f_P(e)}(a)$, $I_{f_P(e)}(d) \leq I_{f_P(e)}(a)$, $F_{f_P(e)}(d) \leq F_{f_P(e)}(a)$;
Further since d is a divisor of a, there exists $t \in \mathbf{Z} - \{0\}$ such that $a = dt$. So $\forall e \in E$,
$T_{f_P(e)}(a) = T_{f_P(e)}(dt) \geq T_{f_P(e)}(d)$, $I_{f_P(e)}(a) = I_{f_P(e)}(dt) \leq I_{f_P(e)}(d)$,
$F_{f_P(e)}(a) = F_{f_P(e)}(dt) \leq F_{f_P(e)}(d)$;
Hence $T_{f_P(e)}(d) = T_{f_P(e)}(a)$, $I_{f_P(e)}(d) = I_{f_P(e)}(a)$, $F_{f_P(e)}(d) = F_{f_P(e)}(a)$, $\forall e \in E$.
Thus for any integer $a(\neq 0)$ there exists a divisor d of n such that $[f_P(e)](d) = [f_P(e)](a)$, $\forall e \in E$.
If $a = 0$ then $T_{f_P(e)}(a) = T_{f_P(e)}(0) = T_{f_P(e)}(n)$, $I_{f_P(e)}(a) = I_{f_P(e)}(0) = I_{f_P(e)}(n)$,
$F_{f_P(e)}(a) = F_{f_P(e)}(0) = F_{f_P(e)}(n)$, $\forall e \in E$.
This follows the theorem.

5.5 *Lemma*

For a neutrosophic soft prime k-ideal N over $(\mathbf{Z}, E)$($\mathbf{Z}$ being the set of integers), $N_0 = p\mathbf{Z}$ is a prime k-ideal of $\mathbf{Z}$ iff p is either zero or prime.

This result is similar to the matter incase of prime ideal in the ring of integers in classical sense. So the proof is omitted.

5.6 Theorem

Let N be a neutrosophic soft prime k-ideal over $(\mathbf{Z}, E)$, $\mathbf{Z}$ being the set of integers. Then $|f_N(e)| = 2$ for each $e \in E$.
Conversely, if N is an NSS over $(\mathbf{Z}, E)$ such that for each $e \in E$, $[f_N(e)](x) = (1, 0, 0)$ when $p|x$ and $[f_N(e)](x) = (\alpha, \beta, \gamma)$ when $p \not| x$, p being a fixed prime and $\beta > 0, \gamma > 0, \alpha < 1$, then N be a neutrosophic soft prime k-ideal over $(\mathbf{Z}, E)$.

Proof. Let N be a neutrosophic soft prime k-ideal over $(\mathbf{Z}, E)$ with $N_0 = p\mathbf{Z}$. By Theorem [5.3], N_0 is a prime k-ideal of $\mathbf{Z}$. Hence by Lemma [5.5], p is prime i.e., p has only two distinct divisors namely $1, p$. So by Theorem [5.4], $|f_N(e)| \leq 2$. But N being a neutrosophic soft prime k-ideal can not be constant, so $|f_N(e)| = 2$, $\forall e \in E$.
Conversely, let N be an NSS over $(\mathbf{Z}, E)$ satisfying the given conditions. Let $x, y \in \mathbf{Z}$.
If $T_{f_N(e)}(x) = \alpha$ or $T_{f_N(e)}(y) = \alpha$ then $T_{f_N(e)}(x+y) = 1$ or α and so
$T_{f_N(e)}(x+y) \geq \min\{T_{f_N(e)}(x), T_{f_N(e)}(y)\}$.
If $T_{f_N(e)}(x) = 1$ and $T_{f_N(e)}(y) = 1$ then $p|x$ and $p|y$. It implies $p|(x+y)$ and
$T_{f_N(e)}(x+y) = 1 = \min\{T_{f_N(e)}(x), T_{f_N(e)}(y)\}$.
Thus in either case $T_{f_N(e)}(x+y) \geq \min\{T_{f_N(e)}(x), T_{f_N(e)}(y)\}$, $\forall x, y \in \mathbf{Z}$, $\forall e \in E$.
Next, if $I_{f_N(e)}(x) = \beta$ or $I_{f_N(e)}(y) = \beta$ then $I_{f_N(e)}(x+y) = 0$ or β and so,
$I_{f_N(e)}(x+y) \leq \max\{I_{f_N(e)}(x), I_{f_N(e)}(y)\}$.
If $I_{f_N(e)}(x) = 0$ and $T_{f_N(e)}(y) = 0$ then $p|x$ and $p|y$. It implies $p|(x+y)$ and
$I_{f_N(e)}(x+y) = 0 = \min\{I_{f_N(e)}(x), I_{f_N(e)}(y)\}$.

Thus in either case $I_{f_N(e)}(x+y) \leq \max\{I_{f_N(e)}(x), I_{f_N(e)}(y)\}$, $\forall x, y \in \mathbf{Z}$, $\forall e \in E$.
Finally, if $F_{f_N(e)}(x) = \beta$ or $F_{f_N(e)}(y) = \beta$ then $F_{f_N(e)}(x+y) = 0$ or β and so $F_{f_N(e)}(x+y) \leq \max\{F_{f_N(e)}(x), F_{f_N(e)}(y)\}$.
If $F_{f_N(e)}(x) = 0$ and $F_{f_N(e)}(y) = 0$ then $p|x$ and $p|y$. It implies $p|(x+y)$ and $F_{f_N(e)}(x+y) = 0 = \min\{F_{f_N(e)}(x), F_{f_N(e)}(y)\}$.
Thus in either case $F_{f_N(e)}(x+y) \leq \max\{F_{f_N(e)}(x), F_{f_N(e)}(y)\}$, $\forall x, y \in \mathbf{Z}$, $\forall e \in E$.
Further if $[f_N(e)](x) = (\alpha, \beta, \gamma)$ then either $[f_N(e)](xy) = (\alpha, \beta, \gamma)$ or $[f_N(e)](xy) = (1, 0, 0)$ i.e., $T_{f_N(e)}(xy) \geq T_{f_N(e)}(x)$, $I_{f_N(e)}(xy) \leq I_{f_N(e)}(x)$, $F_{f_N(e)}(xy) \leq F_{f_N(e)}(x)$.
If $[f_N(e)](x) = (1, 0, 0)$ then $p|x$ and so $p|xy$. Then $[f_N(e)](x) = [f_N(e)](xy) = (1, 0, 0)$. Thus in either case we have $\forall x, y \in \mathbf{Z}$ and $\forall e \in E$,
$T_{f_N(e)}(xy) \geq T_{f_N(e)}(x)$, $I_{f_N(e)}(xy) \leq I_{f_N(e)}(x)$, $F_{f_N(e)}(xy) \leq F_{f_N(e)}(x)$.
So N is a neutrosophic soft ideal over $(\mathbf{Z}, E)$.
We shall now prove that N is a neutrosophic soft k-ideal over $(\mathbf{Z}, E)$.
If $[f_N(e)](x+y) = (\alpha, \beta, \gamma)$ or $[f_N(e)](y) = (\alpha, \beta, \gamma)$, then the inequalities in Definition [5.1] are obvious.
If $[f_N(e)](x+y) = (1, 0, 0)$ or $[f_N(e)](y) = (1, 0, 0)$, then $p|(x+y)$ and $p|y$. It implies $p|x$ and so $[f_N(e)](x) = (1, 0, 0)$. Thus the inequalities in Definition [5.1] hold clearly.
Therefore N is a neutrosophic soft k-ideal over $(\mathbf{Z}, E)$ and so N_0 is a k-ideal over $\mathbf{Z}$.
Finally, we shall prove that N is a neutrosophic soft prime k-ideal over $(\mathbf{Z}, E)$.
To prove it, we shall first show that $N_0 = p\mathbf{Z}$ is a prime k-ideal of $\mathbf{Z}$. Now,
$x \in N_0 \Leftrightarrow [f_N(e)](x) = [f_N(e)](0) = (1, 0, 0) \Leftrightarrow p|x \Leftrightarrow x = pm, m \in \mathbf{Z} \Leftrightarrow x \in p\mathbf{Z}$.
Thus $N_0 = p\mathbf{Z}, p$ being a prime and so N_0 is a prime k-ideal of $\mathbf{Z}$ by Lemma [5.5].
Further, $|f_N(e)| = 2$, $\forall e \in E$ namely $(1, 0, 0)$ and (α, β, γ). So N is not constant over $(\mathbf{Z}, E)$. Now assume two neutrosophic soft ideals S, Q over $(\mathbf{Z}, E)$ such that $SoQ \subseteq N$ and $S \not\subseteq N, Q \not\subseteq N$. Then there exists $x, y \in \mathbf{Z}$ such that $T_{f_S(e)}(x) > T_{f_N(e)}(x)$, $I_{f_S(e)}(x) < I_{f_N(e)}(x)$, $F_{f_S(e)}(x) < F_{f_N(e)}(x)$ and $T_{f_Q(e)}(y) > T_{f_N(e)}(y)$, $I_{f_Q(e)}(y) < I_{f_N(e)}(y)$, $F_{f_Q(e)}(y) < F_{f_N(e)}(y)$, $\forall e \in E$. Then $[f_N(e)](x) = [f_N(e)](y) = (\alpha, \beta, \gamma)$ obviously and so $x, y \notin N_0$. It implies $xy \notin N_0$ as it is a prime k-ideal of an abelian ring $\mathbf{Z}$. So $[f_N(e)](xy) = (\alpha, \beta, \gamma)$. Thus $T_{f_{SoQ}(e)}(xy) \leq T_{f_N(e)}(xy) = \alpha$, $I_{f_{SoQ}(e)}(xy) \geq I_{f_N(e)}(xy) = \beta$, $F_{f_{SoQ}(e)}(xy) \geq F_{f_N(e)}(xy) = \gamma$. But,

$$
\begin{aligned}
&T_{f_{SoQ}(e)}(xy) \geq T_{f_S(e)}(x) * T_{f_Q(e)}(y) > \alpha,\\
&I_{f_{SoQ}(e)}(xy) \leq I_{f_S(e)}(x) \diamond I_{f_Q(e)}(y) < \beta,\\
&F_{f_{SoQ}(e)}(xy) \leq F_{f_S(e)}(x) \diamond F_{f_Q(e)}(y) < \gamma;
\end{aligned}
$$

It opposes the fact. This ends the theorem.

6 Conclusion

The aim of this paper is to put forward the study of the concept neutrosophic soft prime ideal introduced in [26]. Here we have studied about neutrosophic soft completely prime ideal, neutrosophic soft completely semi-prime ideal and neutrosophic soft prime k-ideal. They are defined and illustrated by suitable examples. Their related properties and structural characteristics have been investigated also. Moreover a number of theorems have been developed in virtue of these notions. The concepts

will bring a new opportunity in research and development of algebraic structures over NSS theory context, we expect.

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NSS

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Single Valued Neutrosophic Soft Approach to Rough Sets, Theory and Application

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Abstract. This paper aims to introduce a single valued neutrosophic soft approach to rough sets based on neutrosophic right minimal structure. Some of its properties are deduced and proved. A comparison between traditional rough model and suggested model, by using their properties is concluded to show that Pawlak's approach to rough sets can be viewed as a special case of single valued neutrosophic soft approach to rough sets. Some of rough concepts are redefined and then some properties of these concepts are deduced, proved and illustrated by several examples. Finally, suggested model is applied in a decision making problem, supported with an algorithm.

Keywords: Neutrosophic set, soft set, rough set approximations, neutrosophic soft set, single valued neutrosophic soft set.

1 Introduction

Set theory is a basic branch of a classical mathematics, which requires that all input data must be precise, but almost, real life problems in biology, engineering, economics, environmental science, social science, medical science and many other fields, involve imprecise data. In 1965, L.A. Zadeh [1] introduced the concept of fuzzy logic which extends classical logic by assigning a membership function ranging in degree between 0 and 1 to variables. As a generalization of fuzzy logic, F. Smarandache in 1995, initiated a neutrosophic logic which introduces a new component called indeterminacy and carries more information than fuzzy logic. In it, each proposition is estimated to have three components: the percentage of truth (t %), the percentage of indeterminacy (i %) and the percentage of falsity (f %), his work was published in [2]. From scientific or engineering point of view, neutrosophic set's operators need to be specified. Otherwise, it will be difficult to apply in the real applications. Therefore, Wang et al.[3] defined a single valued neutrosophic set and various properties of it. This thinking is further extended to many applications in decision making problems such as [4, 5].

Rough set theory, proposed by Z. Pawlak [6], is an effective tool in solving many real life problems, based on imprecise data, as it does not need any additional data to discover a knowledge hidden in uncertain data. Recently, many papers have been appeared to development rough set model and then apply it in many real life applications such as [7-11]. In 1999, D. Molodtsov [12], suggested a soft set model. By using it, he created an information system from a collected data. This model has been successfully used in the decision making problems and it has been modified in many papers such as [13-17]. In 2011, F. Feng et al.[18] introduced a soft rough set model and proved its properties. E.A. Marei generalized this model in [19]. In 2013, P.K. Maji [20] introduced neutrosophic soft set, which can be viewed as a new path of thinking to engineers, mathematicians, computer scientists and many others in various tests. In 2014, Broumi et al. [21] introuduced the concept of rough neutrosophic sets. It is generalized and applied in many papers such as [22-31]. In 2015, E.A. Marei [32] introduced the notion of neutrosophic soft rough sets and its modification.

This paper aims to introduce a new approach to soft rough sets based on the neutrosophic logic, named single valued neutrosophic soft (VNS in short) rough set approximations. Properties of VNS-lower and VNS-upper approximations are included along with supported proofs and illustrated examples. A comparison between traditional rough and single valued neutrosophic soft rough approaches is concluded to show that Pawlak's approach to rough sets can be viewed as a special case of single valued neutrosophic soft approach to rough sets. This paper delves into single valued neutrosophic soft rough set by defining some concepts on it as a generalization of rough concepts. Single valued neutrosophic soft rough concepts (NR-concepts in short) include NR-definability, NR-membership function, NR-membership relations, NR-inclusion relations and NR-equality relations. Properties of these concepts are deduced, proved and illustrated by

several examples. Finally, suggested model is applied in a decision making problem, supported with an algorithm.

2 Preliminaries

In this section, we recall some definitions and properties regarding rough set approximations, neutrosophic set, soft set and neutrosophic soft set required in this paper.

Definition 2.1 [6] Lower, upper and boundary approximations of a subset $X \subseteq U$, with respect to an equivalence relation, are defined as

$$\underline{E}(X) = \cup\{[x]_E : [x]_E \subseteq, X\}, \overline{E}(X) = \cup\{[x]_E : [x]_E \cap X \neq \phi\},$$
$$BND_E(X) = \overline{\underline{E}}(X) - \underline{E}(X), where$$
$$[x]_E = \{x' \in U : E(x) = E(x')\}.$$

Definition 2.2 [6] Pawlak determined the degree of crispness of any subset $X \subseteq U$ by a mathematical tool, named the accuracy measure of it, which is defined as

$$\alpha_E(X) = \overline{\underline{E}}(X) / \underline{E}(X), \overline{\underline{E}}(X) \neq \phi.$$

Obviously, $0 \leq \alpha_E(X) \leq 1$. If $\underline{E}(X) = \overline{E}(X)$, then X is crisp (exact) set, with respect to E, otherwise X is rough set.

Properties of Pawlak's approximations are listed in the following proposition.

Proposition 2.1 [6] Let (U, E) be a Pawlak proximation space and let $X, Y \subseteq U$. Then,

(a) $\underline{E}(X) \subseteq X \subseteq \overline{E}(X)$.
(b) $\underline{E}(\phi) = \phi = \overline{E}(\phi)$ and $\underline{E}(U) = U = \overline{E}(U)$.
(c) $\overline{E}(X \cup Y) = \overline{E}(X) \cup \overline{E}(Y)$.
(d) $\underline{E}(X \cap Y) = \underline{E}(X) \cap \underline{E}(Y)$.
(e) $X \subseteq Y$, then $\underline{E}(X) \subseteq \underline{E}(Y)$ and $\overline{E}(X) \subseteq \overline{E}(Y)$.
(f) $\underline{E}(X \cup Y) \supseteq \underline{E}(X) \cup \underline{E}(Y)$.
(g) $\overline{E}(X \cap Y) \subseteq \overline{E}(X) \cap \overline{E}(Y)$.
(h) $\underline{E}(X^c) = [\overline{E}(X)]^c$, X^c is the complement of X.
(i) $\overline{E}(X^c) = [\underline{E}(X)]^c$.
(j) $\underline{E}(\underline{E}(X)) = \overline{E}(\underline{E}(X)) = \underline{E}(X)$.
(k) $\overline{E}(\overline{E}(X)) = \underline{E}(\overline{E}(X)) = \overline{E}(X)$.

Definition 2.3 [33] An information system is a quadruple $IS = (U, A, V, f)$, where U is a non-empty finite set of objects, A is a non-empty finite set of attributes, $V = \cup\{V_e, e \in A\}$, V_e is the value set of attribute e, $f : U \times A \to V$ is called an information (knowledge) function.

Definition 2.4 [12] Let U be an initial universe set, E be a set of parameters, $A \subseteq E$ and let $P(U)$ denotes the power set of U. Then, a pair $S = (F, A)$ is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of U. For $e \in A, F(e)$ may be considered as the set of e-approximate elements of S.

Definition 2.5 [2] A neutrosophic set A on the universe of discourse U is defined as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\}, where$$
$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+, and T, I, F \to \left]^-0,1^+\right[$$

Definition 2.6 [20] Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$, and let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.7 [3] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by truth-embership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(X), I_A(X), F_A(X) \in [0,1]$. When X is continuous, a single valued neutrosophic set A can be written as $A = \int_X (T(x), I(x), F(x))/x, x \in X$. When X is discrete, A can be written as $A = \sum_{i=1}^{n} (T(x_i), I(x_i), F(x_i))/x_i, x_i \in X$.

3 Single valued neutrosophic soft rough set approximations

In this section, we give a definition of a single valued neutrosophic soft (VNS in short) set. VNS-lower and VNS-upper approximations are introduced and their properties are deduced, proved and illustrated by many counter examples.

Definition 3.1 Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$, and let $P(U)$ denotes the set of all single valued neutrosophic sets of U. The collection (G,A) is termed to be VNS set over U, where G is a mapping given by $G : A \to P(U)$.

For more illustration the meaning of VNS set, we consider the following example

Example 3.1 Let U be a set of cars under consideration and E is the set of parameters (or qualities). Each parameter is a neutrosophic word. Consider E = {elegant, trustworthy, sporty, comfortable, modern}. In this case, to define a VNS means to point out elegant cars, trustworthy cars and so on. Suppose that, there are five cars in the universe U, given by $U = \{h_1, h_2, h_3, h_4, h_5\}$ and the set of parameters $A = \{e_1, e_2, e_3, e_4\}$, where $A \subseteq E$ and each e_i is a specific criterion for cars: e_1 stands for elegant, e_2 stands

for trustworthy, e_3 stands for sporty and e_4 stands for comfortable.

A VNS set can be represented in a tabular form as shown in Table 1. In this table, the entries are c_{ij} corresponding to the car h_i and the parameter e_j, where C_{ij} = (true membership value of h_i, indeterminacy-membership value of h_i, falsity membership value of h_i) in $G(e_i)$.

U	e_1	e_2	e_3	e_4
h_1	(.6, .6, .2)	(.8, .4, .3)	(.7, .4, .3)	(.8, .6, .4)
h_2	(.4, .6, .6)	(.6, .2, .4)	(.6, .4, .3)	(.7, .6, .6)
h_3	(.6, .4, .2)	(.8, .1, .3)	(.7, .2, .5)	(.7, .6, .4)
h_4	(.6, .3, .3)	(.8, .2, .2)	(.5, .2, .6)	(.7, .5, .6)
h_5	(.8, .2, .3)	(.8, .3, .2)	(.7, .3, .4)	(.9, .5, .7)

Table1: Tabular representation of (G, A) of Example 3.1.

Definition 3.2 Let (G, A) be a VNS set on a universe U. For any element $h \in U$, a neutrosophic right neighborhood, with respect to $e \in A$ is defined as follows

$$h_e = \{h_i \in U : T_e(h_i) \geq T_e(h), I_e(h_i) \geq I_e(h), F_e(h_i) \leq F_e(h)\}.$$

Definition 3.3 Let (G,A) be a VNS set on U. Neutrosophic right minimal structure is defined as follows

$$\zeta = \{U, \phi, h_e : h \in U, e \in A\}$$

Illustration of Definitions 3.2 and 3.3 is introduced in the following example

Example 3.2 According Example 3.1, we can deduce the following results: $h_{1e_1} = h_{1e_2} = h_{1e_3} = h_{1e_4} = \{h_1\}$, $h_{2e_1} = h_{2e_3} = \{h_1, h_2\}$, $h_{2e_2} = \{h_1, h_2, h_4, h_5\}$, $h_{2e_4} = \{h_1, h_2, h_3\}$, $h_{3e_1} = h_{3e_4} = \{h_1, h_3\}$, $h_{3e_2} = \{h_1, h_3, h_4, h_5\}$, $h_{3e_3} = \{h_1, h_3, h_5\}$, $h_{4e_1} = \{h_1, h_3, h_4\}$, $h_{4e_2} = \{h_4, h_5\}$, $h_{4e_3} = U$, $h_{4e_4} = \{h_1, h_2, h_3, h_4\}$, $h_{5e_1} = h_{5e_2} = h_{5e_4} = \{h_5\}$, $h_{5e_3} = \{h_1, h_5\}$.

It follows that,

$$\zeta = \{\{h_1\}, \{h_5\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_5\}, \{h_4, h_5\}, \{h_1, h_2, h_3\}, \{h_1, h_3, h_4\}, \{h_1, h_3, h_5\}, \{h_1, h_2, h_3, h_4\}, \{h_1, h_2, h_4, h_5\}, \{h_1, h_3, h_4, h_5\}, U, \phi\}$$

Proposition 3.1 Let (G, A) be a VNS set on a universe U, ξ is the family of all neutrosophic right neighborhoods on it, and let

$$R_e : U \to \xi, R_e(h) = h_e$$

Then,

(a) R_e is reflexive relation.

(b) R_e is transitive relation.

(c) R_e may be not symmetric relation.

Proof Let $\langle h_1, T_e(h_1), I_e(h_1), F_e(h_1)\rangle$, $\langle h_2, T_e(h_2), I_e(h_2), F_e(h_2)\rangle$ and $\langle h_3, T_e(h_3), I_e(h_3), F_e(h_3)\rangle \in G(A)$. Then,

(a) Obviously, $T_e(h_1) = T_e(h_1)$, $I_e(h_1) = I_e(h_1)$ and $F_e(h_1) = F_e(h_1)$. For every $e \in A$, $h_1 \in h_{1e}$. Then $h_1 R_e h_1$ and then R_e is reflexive relation.

(b) Let $h_1 R_e h_2$ and $h_2 R_e h_3$, then $h_2 \in h_{1e}$ and $h_3 \in h_{2e}$. Hence, $T_e(h_2) \geq T_e(h_1)$, $I_e(h_2) \geq I_e(h_1)$, $F_e(h_2) \leq F_e(h_1)$, $T_e(h_3) \geq T_e(h_2)$, $I_e(h_3) \geq I_e(h_2)$ and $F_e(h_3) \leq F_e(h_2)$. Consequently, we have $T_e(h_3) \geq T_e(h_1)$, $I_e(h_3) \geq I_e(h_1)$ and $F_e(h_3) \leq F_e(h_1)$. It follows that, $h_3 \in h_{1e}$. Then $h_1 R_e h_3$ and then R_e is transitive relation.

The following example proves (c) of Proposition 3.1.

Example 3.3 From Example 3.2, we have, $h_{1e_1} = \{h_1\}$ and $h_{3e_1} = \{h_1, h_3\}$. Hence, $(h_2, h_1) \in R_{e_1}$ but $(h_1, h_3) \notin R_{e_1}$. Then, R_e isn't symmetric relation.

Definition 3.4 Let (G,A) be a VNS set on U, and let ζ be a neutrosophic right minimal structure on it. Then, VNS-lower and VNS-upper approximations of any subset X based on ζ, respectively, are

$$S_*X = \cup\{Y \in \zeta : Y \subseteq X\},$$
$$S^*X = \cap\{Y \in \zeta : Y \supseteq X\}.$$

Remark 3.1 For any considered set X in a VNS set (G,A), the sets

$$P_{NR}X = S_*X, \quad N_{NR}X = [S^*X]^c,$$
$$b_{NR}X = S^*X - P_{NR}X$$

are called single valued neutrosophic positive, single valued neutrosophic negative and single valued neutrosophic boundary regions of a considered set X, respectively. The real meaning of single valued neutrosophic positive of X is the set of all elements which are surely belonging to X, single valued neutrosophic negative of X is the set of all elements which are surely not belonging to X and single valued neutrosophic boundary of X is the elements of X which are not determined by (G,A). Consequently, the single valued neutrosophic boundary region of any considered set is the initial problem of any real life application.

VNS rough set approximations properties are introduced in the following proposition.

Proposition 3.2 Let (G,A) be a VNS set on U, and let $X, Z \subseteq U$. Then the following properties hold

(a) $S_*X \subseteq X \subseteq S^*X$.

(b) $S_*\varnothing = S^*\varnothing = \varnothing$.

(c) $S_*U = S^*U = U$.

(d) $X \subseteq Z \Rightarrow S_*X \subseteq S_*Z$.

(e) $X \subseteq Z \Longrightarrow S^*X \subseteq S^*Z$.

(f) $S_*(X \cap Z) \subseteq S_*X \cap S_*Z$.

(g) $S_*(X \cup Z) \supseteq S_*X \cup S_*Z$.

(h) $S^*(X \cap Z) \subseteq S^*X \cap S^*Z$

(i) $S^*(X \cup Z) \supseteq S^*X \cup S^*Z$.

Proof

(a) From Definition 3.3, obviously, we can deduce that,

$S_*X \subseteq X \subseteq S^*X$.

(b) From Definition 3.4, we can deduce that $S_*\phi = \phi$ and $S^*\phi = \cap\{Y \in \zeta : Y \supseteq \phi\} = \phi$.

(c) From Property (a), we have $U \subseteq S^*U$ but U is the universe set, then $S^*U \subseteq U$. Also, from Definition 3.4, we have $S_*U = \cup\{Y \in \xi : Y \subseteq U\}$, but $U \in \xi$. Then, $S_*U = U$

(d) Let $X \subseteq Z$ and $h \in S_*X$, then there exists $Y \in \xi$ such that $h \in Y \subseteq X$. But $X \subseteq Z$, then $h \in Y \subseteq Z$. Hence, $h \in S_*Z$. Consequently $S_*X \subseteq S_*Z$.

(e) Let $X \subseteq Z$ and $h \notin S^*Z$. But $S^*Z = \cap\{Y \in \xi : Y \supseteq Z\}$. Then. $h \notin Y$ and $Y \supseteq Z$ such that $U \in \xi$ there exists. But $X \subseteq Z$, then $Y \supseteq X$ and $h \notin Y$. Hence $h \notin S^*Z$. Thus $S^*X \subseteq S^*Z$.

(f) Let $h \in S_*(X \cap Z) = \cup\{Y \in \xi : Y \subseteq X \cap Z\}$. So, there exists $Y \in \xi$ such that, $h \in Y \subseteq X \cap Z$, then $h \in Y \subseteq X$ and $h \in Y \subseteq Z$. Consequently, $h \in S_*X$ and $h \in S_*Z$, then $h \in S_*X \cap S_*Z$. Thus $S_*(X \cap Z) \subseteq S_*X \cap S_*Z$.

(g) Let $h \notin S_*(X \cup Z) = \cup\{Y \in \xi : Y \subseteq X \cup Z\}$. So, for all $Y \in \xi, h \in Y$, we have $Y \not\subset X \cup Z$, then $Y \not\subset X$ and $Y \not\subset Z$. Consequently, $h \notin S_*X$ and $h \notin S_*Z$. So $h \notin S_*X \cup S_*Z$. Thus $S_*(X \cup Z) \supseteq S_*X \cup S_*Z$.

(h) Let $h \notin S^*X \cap S^*Z$. Then, $h \notin S^*X$ or $h \notin S^*Z$ and then there exists $Y \in \xi$ such that $Y \supseteq X, h \notin Y$ or $Y \supseteq X$, $h \notin Y$. Consequently $h \notin S^*(X \cap Z)$. Thus $S^*(X \cap Z) \subseteq S^*X \cap S^*Z$.

(i) Let $h \notin S^*(X \cup Z)$. But $S^*(X \cup Z) = = \cap\{Y \in \xi : Y \supseteq X \cup Z\}$. Then, there exists $Y \in \xi$ such that $Y \supseteq X \cup Z$ and $h \notin Y$. Then, $Y \supseteq X$, $h \notin Y$ and $Y \supseteq Z$, $h \notin Y$. It follows that, $h \notin S^*X \cup S^*Z$. Thus $S^*(X \cup Z) \supseteq S^*X \cup S^*Z$.

The following example illustrates that the converse of Property (a) doesn't hold

Example 3.4 From Example 3.1, if $X = \{h_3\}$, then $S_*X = \phi$. $X \neq S^*X$ and $S_*X \neq X$ Hence. $S^*X = \{h_1, h_3\}$ and

The following example illustrates that the converse of Property (d) doesn't hold

Example 3.5 From Example 3.1, if $X = \{h_2\}$ and $Z = \{h_1, h_2\}$, then $S_*X = \phi$, $S_*Z = \{h_1, h_2\}$. Thus $S_*X \neq S_*Z$.

The following example illustrates that the converse of Property (e) doesn't hold

Example 3.6 From Example 3.1, if $X = \{h_5\}$ and $Z = \{h_2, h_5\}$, then, $S^*X = \{h_5\}$ and $S^*Z = \{h_1, h_2, h_4, h_5\}$. Hence, $S^*X \neq S^*Z$.

The following example illustrates that the converse of Property (f) doesn't hold

Example 3.7 From Example 3.1, If $X = \{h_1, h_3, h_4\}$ and $Z = \{h_1, h_4, h_5\}$, then $S_*X = \{h_1, h_3, h_4\}$, $S_*Z = \{h_1$, $S_*(X \cap Z) \neq S_*X$ Hence. $S_*(X \cap Z) = \{h_1\}$ and $h_4, h_5\}$ $. \cap S_*Z$

The following example illustrates that the converse of Property (g) doesn't hold

Example 3.8 From Example 3.1, if $X = \{h_1\}$ and $Z = \{h_2\}$ then $S_*X = \{h_1\}$, $S_*Z = \phi$ and $S_*(X \cup Z) = \{h_1, h_2\}$. Hence $S_*(X \cup Z) \neq S_*X \cup S_*Z$.

The following example illustrates that the converse of Property (h) doesn't hold

Example 3.9 From Example 3.1, if $X = \{h_1, h_2, h_4\}$ and $Z = \{h_1, h_2, h_5\}$ then $S^*X = \{h_1, h_2, h_4\}$, $S^*Z = \{h_1, h_2, h_4, h_5\}$ and $S^*(X \cap Z) = \{h_1, h_2\}$. Hence $S^*(X \cap Z) \neq S^*X \cap S^*Z$

The following example illustrates that the converse of Property (i) doesn't hold

Example 3.10 From Example 3.1, if $X = \{h_2, h_3\}$ and

$Z = \{h_5\}$ then $S^*X = \{h_1, h_2, h_3\}$, $S^*Z = \{h_5\}$ and $S^*(X \cup Z) = U$. Hence $S^*(X \cup Z) \neq S^*X \cup S^*Z$.

Proposition 3.3 Let (G, A) be a neutrosophic soft set on a unverse U, and let $X, Z \subseteq U$. Then the following properties hold.

(a) $S_*S_*X = S_*X$

(b) $S^*S^*X = S^*X$

(c) $S_*S^*X \subseteq S^*X$

(d) $S^*S_*X \supseteq S_*X$

Proof

(a) Let $W = S_*X$ and $h \in W = \cup\{Y \in \zeta : Y \subseteq X\}$. Then, for some $e \in A$, we have $h \in Y \subseteq W$. So $h \in S_*W$. Hence $W \subseteq S_*W$. Thus, $S_*W \subseteq S_*S_*W$. Also, from Property (a) of Proposition 3.2, we have $S_*X \subseteq X$ and by using Property (d) of Proposition 3.2, we get $S_*S_*X \subseteq S_*X$. Consequently. $S_*X = S_*S_*X$

(b) Let $W = S^*X$ and $h \notin W$, from Definition 3.4, we have $W = \cap\{Y \in \xi : Y \supseteq X\}$. Then there exists $Y \in \xi$, such that $Y \supseteq X$ and $h \notin Y$. Hence, there exists $Y \in \xi$, such that $Y \supseteq W$ and $h \notin Y$, it follows that $h \notin S^*W$. Consequently $W \supseteq S^*W$. Also, by using Property (a) of Proposition 3.2, we have $W \subseteq S^*W$. Thus $S^*S^*W = S^*W$

Properties (c) and (d) can be proved directly from Proposition 3.2.

The following example illustrates that the converse of Property (c) doesn't hold.

Example 3.11 From Example 3.1, if $X = \{h_4\}$. Then $S^*X = \{h_4\}$ and $S_*S^*X = \phi$. Hence, $S_*S^*X \neq S^*X$.

The following example illustrates that the converse of Property (c) doesn't hold.

Example 3.12 From Example 3.1, if $X = \{h_1, h_2, h_5\}$, then $S_*X = \{h_1, h_2, h_5\}$ and $S^*S_*X = \{h_1, h_2, h_4, h_5\}$. Hence $S^*S_*X \neq S_*X$

Proposition 3.4 Let (G, A) be a VNS set on U and let $X, Z \subseteq U$. Then

$$S_*(X - Z) \subseteq S_*X - S_*Z$$

Proof

Let $h \in S_*(X - Z) = \cup\{Y \in \xi : Y \subseteq (X - Z)\}$. So, there exists $Y \in \xi$ such that $h \in Y \subseteq (X - Z)$, then $h \in Y \subseteq X$ and $h \in Y \not\subset Z$. Consequently, $h \in S_*X$ and $h \notin S_*Z$, then $h \in S_*X - S_*Z$. Therefore $S_*(X - Z) \subseteq S_*X - S_*Z$.

The following example illustrates that the converse of Proposition 3.4 doesn't hold.

Example 3.13 From Example 3.1, if $X = \{h_1, h_3, h_5\}$ and $Z = \{h_1, h_5\}$, then $S_*X = \{h_1, h_3, h_5\}$, $S_*Z = \{h_1, h_5\}$, $S_*(X - Z) = \phi$ and $S_*X - S_*Z = \{h_3\}$. Hence, $S_*(X - Z) \neq S_*X - S_*Z$

Proposition 3.5 Let (G, A) be a VNS set on U and let $X, Z \subseteq U$. Then the following properties don't hold

(a) $S_*X^c = [S^*X]^c$

(b) $S^*X^c = [S_*X]^c$

(c) $S^*(X - Z) = S^*X - S^*Z$

The following example proves Properties (a) and (b) of Proposition 3.5.

Example 3.14 From Example 3.1, if $X = \{h_1\}$. Then, $S_*X = S^*X = \{h_1\}$, $S_*X^c = \{h_4, h_5\}$ and $S^*X^c = U$. Thus $S_*X^c \neq [S^*X]^c$ and $S^*X^c \neq [S_*X]^c$

The following example proves Property (c) of Proposition 3.5.

Example 3.15 From Example 3.1, if $X = \{h_1, h_2\}$ and $Z = \{h_1\}$. Then $S^*X = \{h_1, h_2\}$, $S^*Z = \{h_1\}$, $S^*(X - Z) = \{h_1, h_2\}$. Hence $S^*(X - Z) \neq S^*X - S^*Z$.

Remark 3.2 A comparison between traditional rough and single valued neutrosophic soft rough approaches, by using their properties, is concluded in Table 2, as follows

4 Single valued neutrosophic soft rough concepts

In this section, some of single valued neutrosophic soft rough concepts (NR-concepts in short) are defined as a generalization of traditional rough concepts.

Definition 4.1 Let (G, A) be a VNS set on U. A subset $X \subseteq U$ is called

(a) NR-definable (NR-exact) set if $S_*X = S^*X = X$

(b) Internally NR-definable set if $S_*X = X$ and $S^*X \neq X$

(c) Externally NR-definable set if $S_*X \neq X$ and $S^*X = X$

(d) NR-rough set if $S_*X \neq X$ and $S^*X \neq X$

The following example illustrates Definition 4.1.

Example 4.1 From Example 3.1, we can deduce that $\{h_1\}$, $\{h_5\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_5\}, \{h_4, h_5\}$, $\{h_1, h_2, h_3\}, \{h_1, h_3, h_4\}, \{h_1, h_3, h_5\}, \{h_1, h_4, h_5\}, \{h_1, h_2, h_3, h_4\}, \{h_1, h_2, h_4, h_5\}, \{h_1, h_3, h_4, h_5\}$ are NR-definable sets, $\{h_1, h_2, h_5\}, \{h_1, h_2, h_3, h_5\}$ are internally NR-definable sets, $\{h_4\}, \{h_1, h_4\}, \{h_1, h_2, h_4\}$ are externally NR-definable sets and the rest of proper subsets of U are

NR-rough sets.

We can determine the degree of single valued neutrosophic soft-crispness (exactness) of any subset $X \subseteq U$ by using NR-accuracy measure, denoted by $C_* X$, which is defined as follows

Definition 4.2 Let (G,A) be a VNS on U, and let $X \subseteq U$. Then

$$C_* X = S_* X / S^* X, X \neq \phi$$

Remark 4.1 Let (G,A) be a VNS on U. A subset $X \subseteq U$ is NR-definable (NR-exact) if and only if $C_* X = 1$.

Definition 4.3 Let (G,A) be a VNS on U and let $X \subseteq U$, $x \notin X$. NR-membership function of an element x to a set X denoted by $\mu_X x$ is defined as follows:
$\mu_X x = | x_A \cap X | / | x_A |$, where $x_A = \cap \{x_e : e \in A\}$ and x_e is a neutrosophic right neighborhood, defined in Definition 3.2.

Proposition 4.1 Let (G,A) be a VNS on U, $X \subseteq U$ and let $\mu_X x$ be the membership function defined in Definition 4.3. Then

$$\mu_X x \in [0,1]$$

Proof

Where $\phi \subseteq x_A \cap X \subseteq x_A$ then $0 \leq |x_A \cap X| \leq |x_A|$ and then $0 \leq \mu_X x \leq 1$.

Proposition 4.2 Let (G,A) be a VNS on U and let $X \subseteq U$, then

$$\mu_X x = 1 \Rightarrow x \in X$$

Proof

Let $\mu_X x = 1$, then $|x_A \cap X| = |x_A|$. Consequantly $x_A \subseteq X$. From Proposition 3.1, we have R_e is a reflexive relation for all $e \in A$. Hence $x \in x_e \forall e \in A$. It follows that $x \in x_A$. Thus $x \in X$

The following example illustrates that the converse of Proposition 4.2 doesn't hold.

Example 4.2 From Example 3.2, we get $h_{3A} = \{h_1, h_3\}$. If $X = \{h_2, h_3, h_5\}$, then $\mu_X h_3 = 1/2$. Although $h_3 \in X$

Proposition 4.3 Let (G,A) be a VNS on U and let $X, Z \subseteq U$. If $X \subseteq Z$, then the following properties hold

(a) $\mu_X x \leq \mu_Z x$

(b) $\mu_{S_* X} x \leq \mu_{S_* Z} x$

(c) $\mu_{S^* X} x \leq \mu_{S^* Z} x$

Proof

(a) Where $X \subseteq U$, for any $x \subseteq U$ we can deduce that $\mu_X x \leq \mu_Z x$. Thus $|x_A \cap X| \leq |x_A \cap Z|$ then $\subseteq x_A \cap Z$, $x_A \cap X$

We get the proof of Properties (b) and (c) of Proposition 4.3, directly from property (a) of Proposition 4.3 and properties (d) and (e) of Proposition 3.2.

Traditional rough properties	VNS rough properties
$\overline{E}(X \cup Z) = \overline{E}X \cup \overline{E}Z$	$S^*(X \cup Z) \supseteq S^* X \cup S^* Z$
$\underline{E}(X \cap Y) = \underline{E}(X) \cap \underline{E}(Y)$	$S_*(X \cap Z) \subseteq S_* X \cap S_* Z$
$\underline{E}(\overline{E}(X)) = \overline{E}(X)$	$S_* S^* X \subseteq S^* X$
$\overline{E}(\underline{E}(X)) = \underline{E}(X)$	$S^* S_* X \supseteq S_* X$
$\underline{E}(X^C) = [\overline{E}(X)]^C$	$S_* X^c \neq [S^* X]^c$
$\overline{E}(X^c) = [\underline{E}(X)]^c$	$S^* X^c \neq [S_* X]^c$

Table 2: Comparison between traditional, VNS rough

Proposition 4.4 Let (G,A) be a VNS on U and let $X \subseteq U$, then the following properties hold

(a) $\mu_{S_* X} x \leq \mu_X x$

(b) $\mu_X x \leq \mu_{S^* X} x$

(c) $\mu_{S_* X} x \leq \mu_{S^* X} x$

Proof can be obtained directly from Propositions 3.2 and property (a) of Proposition 4.3.

Definition 4.4 Let (G,A) be a VNS set on U, and let $x \in U$, $X \subseteq U$. NR-membership relations, denoted by $\in_*$ and $\in^*$ are defined as follows

$$x \in_* X \text{ if } x \in S_* X \text{ and } x \in^* X \text{ if } x \in S^* X$$

Proposition 4.5 Let (G,A) be a VNS set on U, and let $x \in U$, $X \subseteq U$. Then

(a) $x \in_* X \Rightarrow x \in X$

(b) $x \notin^* X \Rightarrow x \notin X$

Proof

(a) Let $x \in_* X$, hence by using Definition 4.4, we get $x \in S_* X$.
But from Proposition 3.2, we have $S_* X \subseteq X$, then $x \in X$.

(b) Let $x \in X$, according to Proposition 3.2, we have $X \subseteq S^* X$, then $x \in S^* X$, by using Definition 4.4, we can deduce that $x \in^* X$.
Consequently $x \notin^* X \Rightarrow x \notin X$.

The following example illustrates that the converse of Proposition 4.5 doesn't hold.

Example 4.3 From Example 3.1, if $X = \{h_2, h_5\}$, then $S_* X = \{h_5\}$ and $S^* X = \{h_1, h_2, h_4, h_5\}$. Hence, $h_2 \notin_* X$, although $h_2 \in X$ and $h_4 \notin X$, although $h_4 \in^* X$.

Proposition 4.6 Let (G,A) be a VNS on U and let $X \subseteq U$. Then the following properties hold

(a) $x \in_* X \Rightarrow \mu_X x = 1$

(b) $\mu_X x = 1 \Rightarrow x \in^* X$

Proof can be obtained directly from Definition 4.4 and Propositions 4.2 and 4.5.

The following example illustrates that the converse of property (a) does not hold.

Example 4.4 From Example 3.1, if $X = \{h_1, h_4\}$ then $S_*X = \{h_1\}$ and $h_{4A} = \{h_4\}$, it follows that $\mu_X h_4 = 1$. Although $h_4 \notin_* X$

The following example illustrates that the converse of property (b) does not hold.

Example 4.5 From Example 3.1, if $X = \{h_2\}$, then $S^*X = \{h_1, h_2\}$ and $h_{2A} = \{h_1, h_2\}$, it follows that $h_2 \in^* X$, although $\mu_X h_2 \neq 1$

Proposition 4.7 Let (G,A) be a VNS on U and let $X \subseteq U$. Then

(a) $\mu_X x = 0 \Rightarrow x \notin X$

(b) $\mu_X x = 0 \Rightarrow x \notin_* X$

Proof is straightforward and therefore is omitted.

The following example illustrates that the converse of property (a), does not hold.

Example 4.6 From Example 3.1, if $X = \{h_1, h_3, h_4\}$ and from Example 3.2, we get $h_{2A} = \{h_1, h_2\}$, then $\mu_X h_2 \neq 0$, although $h_2 \notin X$

The following example illustrates that the converse of property (b), does not hold.

Example 4.7 From Example 3.1, if $X = \{h_1, h_4, h_5\}$, then $S_*X = \{h_1, h_4, h_5\}$, from Example 3.2, we get $h_{2A} = \{h_1, h_2\}$, it follows that $\mu_X h_2 \neq 0$, although $h_2 \notin_* X$

Proposition 4.8 Let (*G*,*A*) be a VNS on *U* and let *X*⊆*U*. The following property does not hold

$$\mu_X x = 0 \Rightarrow x \notin^* X$$

The following example proves Proposition 4.8.

Example 4.8 From Example 3.1, if $X = \{h_2\}$ then $S^*X = \{h_1, h_2\}$, from Example 3.2, we get $h_{1A} = \{h_1\}$, it follows that $h_1 \in^* X$, although $\mu_X h_1 = 0$

Definition 4.5 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. NR-inclusion relations, denoted by $\subset_*$ and $\subset^*$ which are defined as follows

$$X \subset_* Z \text{ If } \quad S_*X \subseteq S_*Z$$

$$X \subset^* Z \text{ If } \quad S^*X \subseteq S^*Z$$

Proposition 4.9 Let (G,A) be a VNS on U and let $X, Z \subseteq U$. Then

$$X \subseteq Z \Rightarrow X \subset_* Z \wedge X \subset^* Z$$

Proof comes directly From Proposition 3.2.

The following example illustrates that, the converse of Proposition 4.9 doesn't hold.

Example 4.9 In Example 3.1, if $X = \{h_1, h_4\}$ and $Z = \{h_1, h_2, h_5\}$, then $S_*X = \{h_1\}$, $S_*Z = \{h_1, h_2, h_5\}$, $S^*X = \{h_1, h_4\}$ and $S^*Z = \{h_1, h_2, h_4, h_5\}$. Hence, $X \subset_* Z$ and $X \subset^* Z$. Although $X \not\subset Z$

From Definition 4.5 and Proposition 4.3, the following remarks can be deduced

Remark 4.2 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. If $X \subset_* Z$, then the following properties hold

(a) $\mu_{S_*X} x \leq \mu_{S_*Z} x$

(b) $\mu_{S_*X} x \leq \mu_Z x$

(c) $\mu_{S_*X} x \leq \mu_{S^*Z} x$

Remark 4.3 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. If $X \subset^* Z$, then the following properties hold

(a) $\mu_{S^*X} x \leq \mu_{S^*Z} x$

(b) $\mu_X x \leq \mu_{S^*Z} x$

(c) $\mu_{S_*X} x \leq \mu_{S^*Z} x$

Definition 4.6 Let (G,A) be a VNS on U and let $X, Z \subseteq U$. NR-equality relations are defined as follows

$$X =_* Z \text{ If } \quad S_*X = S_*Z$$

$$X =^* Z \text{ If } \quad S^*X = S^*Z$$

$$\text{If } X =^*_* Z \quad X =_* Z \wedge X =^* Z$$

The following example illustrates Definition 4.6.

Example 4.10 According to Example 3.1. Let $A = \{e_1\}$, then $\xi = \{U, \phi, \{h_1\}, \{h_5\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_3, h_4\}\}$. If $X_1 = \{h_2\}$, $X_2 = \{h_3\}, X_3 = \{h_1, h_2\}, X_4 = \{h_2, h_3\}$ and $X_5 = \{h_2, h_4\}$, then $S_*X_1 = S_*X_2 = \phi$, $S^*X_1 = S^*X_3 = \{h_1, h_2\}$, $S_*X_4 = S_*X_5 = \phi$ and $S^*X_4 = S^*X_5 = U$. Consequently $X_1 =_* X_2$, $X_1 =^* X_3$ and $X_4 =^*_* X_5$

Proposition 4.10 Let (*G*,*A*) be a VNS set on U and let $X, Z \subseteq U$. Then

(a) $X =_* S_*X$

(b) $X =^* S^*X$

(c) $X = Z \Rightarrow X =^*_* Z$

(d) $X \subseteq Z, Z =_* \phi \Rightarrow X =_* \phi$

(e) $X \subseteq Z, X =_* U \Rightarrow Z = U$

(f) $X \subseteq Z, Z =^* \phi \Rightarrow X = \phi$

(g) $X \subseteq Z, X =^* U \Rightarrow Z =^* U$

Proof. From Definition 4.6 and Propositions 3.2 and 3.3 we get the proof, directly.

From Definition 4.6 and Proposition 4.3, the following remarks can be deduced

Remark 4.4 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. If $X =_* Z$, then the following properties hold

(a) $\mu_{S_*X} x = \mu_{S_*Z} x$

(b) $\mu_{S_*X} x \leq \mu_Z x$

(c) $\mu_{S_*X} x \leq \mu_{S^*Z} x$

Remark 4.5 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. If $X =^* Z$**,** then the following properties hold

(a) $\mu_{S^*X} x \leq \mu_{S^*Z} x$

(b) $\mu_X x \leq \mu_{S^*Z} x$

(c) $\mu_{S_*X} x \leq \mu_{S^*Z} x$

The following remark is introduced to show that Pawlak's approach to rough sets can be viewed as a special case of proposed model.

Remark 4.6 Let (*G*,*A*) be a VNS on U and let $X, Z \subseteq U$. If we consider the following case

(If $T_e(h_i) \geq 0.5$, then $e(h) = 1$, otherwise $e(h) = 0$)

and the neutrosophic right neighborhood of an element *h* is replaced by the following equivalence class

$$[h]_e = \{h_i \in U : e(h_i) = e(h), e \in A\}.$$

Then VNS-lower and VNS-upper approximations will be traditional Pawlak's approximations. It follows that NR-concepts will be Pawlak's concepts. Therefor Pawlak's approach to rough sets can be viewed as a special case of suggested single valued neutrosophic soft approach to rough sets.

5 A decision making problem

In this section, suggested single valued neutrosophic soft rough model is applied in a decision making problem. We consider the problem to select the most suitable car which a person X is going to choose from n cars $(h_1, h_2, ..., h_n)$ by using m parameters $(e_1, e_2, .., e_m)$.
Since these data are not crisp but neutrosophic, the selection is not straightforward. Hence our problem in this section is to select the most suitable car with the choice parameters of the person *X*. To solve this problem, we need the following definitions

Definition 5.1 Let (*G*,*A*) be a VNS set on $U = \{h_1, h_2, ..., h_n\}$ as the objects and $A = \{e_1, e_2, .., e_m\}$ is the set of parameters. The value matrix is a matrix whose rows are labeled by the objects, its columns are labeled by the parameters and the entries C_{ij} are calculated by

$$C_{ij} = (T_{ej}(h_i) + I_{ej}(h_i) - F_{ej}(h_i)), \ 1 \leq i \leq n, 1 \leq j \leq m$$

Definition 5.2 Let (*G*,*A*) be a VNS set on $U = \{h_1, h_2, ..., h_n\}$, where $A = \{e_1, e_2, .., e_m\}$. The score of an object h_j is defined as follows

$$S(h_i) = \Sigma_{j=1}^{m} C_{ij}$$

Remark 5.1 Let (*G*, *A*) be a VNS set on U and $A = \{e_1, e_2, .., e_m\}$ then is the set of parameters.

(a) $-1 \leq C_{ij} \leq 2, \ 1 \leq i \leq n, 1 \leq j \leq m$

(b) $-m \leq S(h_i) \leq 2m, \ h_i \in U$

The real meaning of C_*A is the degree of crispness of A. Hence, if $C_*A = 1$, then A is *NR*-definable set. It means that the collected data are sufficient to determine the set A. Also, from the meaning of the neutrosophic right neighborhood, we can deduce the most suitable choice by using the following algorithm.

Algorithm

1. Input VNS set (*G*,*A*)
2. Compute the accuracy measures of all singleton sets
3. Consider the objects of *NR*-definable singleton sets
4. Compute the value matrix of the considered objects
5. Compute the score of all considered objects in a tabular form
6. Find the maximum score of the considered objects
7. If there are more than one object has the maximum scare, then any object of them could be the suitable choice
8. If there is no *NR*-definable singleton set, then we consider the objects of all *NR*-definable sets consisting two elements and then repeat steps (4-7), else, consider the objects of all *NR*-definable sets consisting three elements and then repeat steps (4-7),and so on...

For illustration the previous technique, the following example is introduced.

Example 5.1 According to Example 3.1, we can create Tables 3, as follows

Singleton sets	$\{h_1\}$	$\{h_2\}$	$\{h_3\}$	$\{h_4\}$	$\{h_5\}$
C_*X	1	0	0	0	1

Table 3: Accuracy measures of all singleton sets.

Hence $C_*\{h_1\} = C_*\{h_5\} = 1$. It follows that h_1 and h_5 are the *NR*-definable singleton sets. Consequently h_1 and h_5 are concidered objects. Therefore Table 4 can be created as follows

Object	e_1	e_2	e_3	e_4
h_1	(.6,.6,.2)	(.8,.4,.3)	(.7,.4,.3)	(.8,.6,.4)
h_5	(.8,.2,.3)	(.8,.3,.2)	(.7,.3,.4)	(.9,.5,.7)

Table 4: Tabular representation of considered objects.

The value matrix of considered objects can be viewed as Table 5.

Object	e_1	e_2	e_3	e_4
h_1	1	0.9	0.8	1
h_5	0.7	0.9	0.6	0.7

Table 5: Value matrix of considered objects.

Finally, the scores of considered objects are concluded in Table 6, as follows

Object	Score of the object
h_1	3.7
h_5	2.9

Table 6: The scores of considered objects.

Clearly, the maximum score is 3.7, which is scored by the car h_1. Hence, our decision in this case study is that a car h_1 is the most suitable car for a person X, under his choice parameters. Also, the second suitable car for him is a car h_5.

Obviously, the selection is dependent on the choice parameters of the buyer. Consequently, the most suitable car for a person X need not be suitable car for another person Y.

Conclusion

This paper introduces the notion of single valued neutrosophic soft rough set approximations by using a new neighborhood named neutrosophic right neighborhood. Suggested model is more realistic than the other traditional models, as each proposition is estimated to have three components: the percentage of truth, the percentage of indeterminacy and the percentage of falsity. Several properties of single valued neutrosophic soft rough sets have been defined and propositions and illustrative examples have been presented. It has been shown that Pawlak's approach to rough sets can be viewed as a special case of single valued neutrosophic soft approach to rough sets. Finally, proposed model is applied in a decision making problem, supported with algorithm.

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A novel approach to nano topology via neutrosophic sets

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Abstract: The main objective of this study is to introduce a new hybrid intelligent structure called Neutrosophic nano topology. Fuzzy nano topology and intuitionistic nano topology can also be deduced from the neutrosophic nano topology. Based on the neutrosophic nano approximations we have classified neutrosophic nano topology. Some properties like neutrosophic nano interior and neutrosophic nano closure are derived.
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1 INTRODUCTION

Nano topology explored by Thivagar et.al can be described as a collection of nano approximations, a non-empty finite universe and empty set for which equivalence classes are buliding blocks. It is named as nano topology because whatever may be the size of the universe it has at most five open sets. After this, there has been many models built upon different aspect, i.e, universe, relations, object and operators. One of the interesting generalizations of the theories of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F.Smarandache. Neutrosophic set is described by three functions : a membership function, indeterminacy function and a nonmembership function that are independently related. The theories of neutrosophic set have achieved greater success in various areas such as medical diagnosis, database, topology, image processing and decision making problem. While the neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data, the theory of rough set is a powerful mathematical tool to deal with incompleteness. Neutrosophic sets and rough sets are two different topics, none conflicts the other. The main objective of this study is to introduce a new hybrid intelligent structure called neutrosophic nano topology. The significance of introducing hybrid structures is that the computational techniques, based on any one of these structures alone, will not always yield the best results but a fusion of two or more of them can often give better results. The rest of this paper is organized as follows. Some preliminary concepts required in our work are briefly recalled in section 2. In section 3 , the concept of neutrosophic nano topology is investigated. Section 4 concludes the paper with some properties on neutrosophic nano interior and neutrosophic nano closure.

2 Preliminaries

The following recalls requisite ideas and preliminaries necessitated in the sequel of our work.

Definition 2.1 *[8]*: Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in \mathcal{U}} \{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in \mathcal{U}} \{R(x) : R(x) \cap X \neq \phi\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Remark 2.2 *[8]*: If $(\mathcal{U}, R)$ is an approximation space and X, $Y \subseteq \mathcal{U}$, then the following statements hold:

(i) $L_R(X) \subseteq X \subseteq U_R(X)$.

(ii) $L_R(\phi) = U_R(\phi) = \phi$ and $L_R(\mathcal{U}) = U_R(\mathcal{U}) = \mathcal{U}$.

(iii) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$.

(iv) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$

(v) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.

(vi) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$.

(vii) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$, whenever $X \subseteq Y$.

(viii) $U_R(X^C) = [L_R(X)]^C$ and $L_R(X^C) = [U_R(X)]^C$.

(ix) $U_R U_R(X) = L_R U_R(X) = U_R(X)$.

(x) $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.3 *[8]*: Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$ where X $\subseteq \mathcal{U}$. $\tau_R(X)$ satisfies the following axioms:

(i) $\mathcal{U}$ and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on $\mathcal{U}$ called the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets.

Proposition 2.4 *[8]*: Let $\mathcal{U}$ be a non-empty finite universe and $X \subseteq \mathcal{U}$. Then the following statements hold:

(i) If $L_R(X) = \phi$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi\}$, is the indiscrete nano topology on $\mathcal{U}$.

(ii) If $L_R(X) = U_R(X) = X$, then the nano topology, $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X)\}$.

(iii) If $L_R(X) = \phi$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, U_R(X)\}$.

(iv) If $L_R(X) \neq \phi$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), B_R(X)\}$.

(v) If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \phi$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \phi, L_R(X), U_R(X), B_R(X)\}$ is the discrete nano topology on $\mathcal{U}$.

Definition 2.5 *[3]*: Let X be a non empty set. A fuzzy set A is an object having the form A $= \{< x : \mu_A(x), x \in X\}$, where $0 \leq \mu_A(x) \leq 1$ represent the degree of membership of each $x \in X$ to the set A.

Definition 2.6 *[2]*: Let X be a non empty set. An intuitionstic set A is of the form A $= \{< x : \mu_A(x), \nu_A(x), x \in X\}$, where $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership function and the degree of non membership respectively of each $x \in X$ to the set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Definition 2.7 *[6]*: Let X be an universe of discourse with a generic element in X denoted by x, the neutrosophic set is an object having the form A $= \{< x : \mu_A(x), \sigma_A(x), \nu_A(x) >, x \in X\}$, where the functions $\mu, \sigma, \nu : X \to [0, 1]$ define respectively the degree of membership or truth , the degree of indeterminancy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set A with the condition. $-0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$.

3 Neutrosophic Nano Topological Space

In this section we introduce the notion of neutrosophic nano topology by means of nano neutrosophic nano approximations namely neutrosophic nano lower, neutrosophic nano upper and neutrosophic nano boundary. From Neutrosophic nano topology we have also defined and deduced intuitionistic nano topology and fuzzy nano topology.

Definition 3.1 : Let $\mathcal{U}$ be a non-empty set and R be an equivalence relation on $\mathcal{U}$. Let F be a neutrosophic set in $\mathcal{U}$ with the membership function μ_F, the indeterminancy function σ_F and the non-membership function ν_F. The neutrosophic nano lower, neutrosophic nano upper approximation and neutrosophic nano boundary of F in the approximation $(\mathcal{U}, R)$ denoted by $\underline{N}(F), \overline{N}(F) and\ BN(F)$ are respectively defined as follows:

(i) $\underline{N}(F) = \{< x, \mu_{\underline{R}(A)}(x), \sigma_{\underline{R}(A)}(x), \nu_{\underline{R}(A)}(x) > / y \in [x]_R, x \in \mathcal{U}\}$.

(ii) $\overline{N}(F) = \{< x, \mu_{\overline{R}(A)}(x), \sigma_{\overline{R}(A)}(x), \nu_{\overline{R}(A)}(x) > / y \in [x]_R, x \in \mathcal{U}\}$.

(iii) BN(F)$= \overline{N}(F) - \underline{N}(F)$.

where $\mu_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \mu_A(y)$, $\sigma_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \sigma_A(y)$, $\nu_{\underline{R}(A)}(x) = \bigvee_{y\in[x]_R} \nu_A(y)$.

$\mu_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \mu_A(y)$, $\sigma_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \sigma_A(y)$, $\nu_{\overline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \nu_A(y)$.

Definition 3.2 : Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and F be a neutrosophic set in $\mathcal{U}$ and if the collection $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), BN(F)\}$ forms a topology then it is said to be a neutrosophic nano topology. We call $(\mathcal{U}, \tau_N(F))$ as the neutrosophic nano topological space. The elements of $\tau_N(F)$ are called neutrosophic nano open sets.

Remark 3.3 : From **Neutrosophic nano topology** we can deduce and define the **fuzzy nano topology** and **intuitionistic nano topology**. Fuzzy nano topology is obtained by considering the membership values alone whereas in case of intuitionistic nano topology both membership and non member ship values are considered.

Definition 3.4 : Let $\mathcal{U}$ be a non-empty set and R be an equivalence relation on $\mathcal{U}$. Let F be an intuitionistic set in $\mathcal{U}$ with the membership function μ_F and the non-membership function ν_F. The intuitionistic nano lower, intuitionistic nano upper approximation and intuitionistic nano boundary of F in the approximation $(\mathcal{U}, R)$ denoted by $\underline{I}(F), \overline{I}(F) and\ B_I(F)$ are respectively defined as follows:

(i) $\underline{I}(F) = \{< x, \mu_{\underline{R}(A)}(x), \nu_{\underline{R}(A)}(x) > /y \in [x]_R, x \in \mathcal{U}\}$.

(ii) $\overline{I}(F) = \{< x, \mu_{\overline{R}(A)}(x), \nu_{\overline{R}(A)}(x) > /y \in [x]_R, x \in \mathcal{U}\}$.

(iii) $B_I(F) = \overline{I}(F) - \underline{I}(F)$.

where $\mu_{\underline{R_{\mathcal{I}}}(A)}(x) = \bigwedge_{y\in[x]_R} \mu_A(y)$, $\nu_{\underline{R_{\mathcal{I}}}(A)}(x) = \bigvee_{y\in[x]_R} \nu_A(y)$.

$\mu_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \mu_A(y)$, $\nu_{\overline{R_{\mathcal{I}}}(A)}(x) = \bigwedge_{y\in[x]_R} \nu_A(y)$.

Definition 3.5 : Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and F be an intuitionistic set in $\mathcal{U}$ and if the collection $\tau_I(F) = \{0_N, 1_N, \underline{I}(F), \overline{I}(F), B_I(F)\}$ forms a topology then it is said to be a intuitionistic nano topology. We call $(\mathcal{U}, \tau_I(F))$ as the intuitionistic nano topological space. The elements of $\tau_I(F)$ are called intuitionistic nano open sets.

Definition 3.6 : Let $\mathcal{U}$ be a non-empty set and R be an equivalence relation on $\mathcal{U}$. Let F be a fuzzy set in $\mathcal{U}$ with the membership function μ_F. Then the fuzzy nano lower, fuzzy nano upper approximation of F and fuzzy nano boundary of F in the approximation $(\mathcal{U}, R)$ denoted by $\underline{\mathcal{F}}(F), \overline{\mathcal{F}}(F) and\ B_{\mathcal{F}}(F)$ are respectively defined as follows:

(i) $\underline{\mathcal{F}}(F) = \{< x, \mu_{\underline{R}(A)}(x) > /y \in [x]_R, x \in \mathcal{U}\}$.

(ii) $\overline{\mathcal{F}}(F) = \{< x, \mu_{\overline{R}(A)}(x) > /y \in [x]_R, x \in \mathcal{U}\}$.

(iii) $B_{\mathcal{F}}(F) = \overline{\mathcal{F}}(F) - \underline{\mathcal{F}}(F)$.

where $\mu_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \mu_A(y)$, $\mu_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \mu_A(y)$

Definition 3.7 : Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and F be a fuzzy set in $\mathcal{U}$ and if the collection $\tau_{\mathcal{F}}(F) = \{0_N, 1_N, \underline{\mathcal{F}}(F), \overline{\mathcal{F}}(F), B_{\mathcal{F}}(F)\}$ forms a topology then it is said to be a fuzzy nano topology. We call $(\mathcal{U}, \tau_{\mathcal{F}}(F))$ as the fuzzy nano topological space. The elements of $\tau_{\mathcal{F}}(F)$ are called fuzzy nano open sets.

Remark 3.8 : Thus from the above definitions of intuitionistic and fuzzy nano topologies we can assure that throughout this paper all the properties and examples also holds good when it is possible for neutrosophic nano topology.

Remark 3.9 : Since our main purpose is to construct tools for developing neutrosophic nano topological spaces, we must introduce 0_N , 1_N and certain neutrosophic set operations in X as follows:

Definition 3.10 : Let $\mathcal{U}$ be a nonempty set and the neutrosophic sets A and B in the form A = $\{< x : \mu_A(x), \sigma_A(x), \nu_A(x) >, x \in \mathcal{U}\}$, B = $\{< x : \mu_B(x), \sigma_B(x), \nu_B(x) >, x \in \mathcal{U}\}$. Then the following statements hold:

(i) $0_N = \{< x, 0, 0, 1 >: x \in \mathcal{U}\}$ and $1_N = \{< x, 1, 1, 0 >: x \in \mathcal{U}\}$.

(ii) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \nu_A(x) \geq \nu_B(x) for\ all\ x \in \mathcal{U}\}$.

(iii) A = B iff $A \subseteq B$and $B \subseteq A$.

(iv) $A^C = \{< x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x) >: x \in \mathcal{U}\}$.

(v) $A \cap B = \{x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) for\ all\ x \in \mathcal{U}\}$.

(vi) $A \cup B = \{x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x) for\ all\ x \in \mathcal{U}\}$.

Theorem 3.11 *[8]*: Let $\mathcal{U}$ be a non-empty finite universe and $X \subseteq \mathcal{U}$. Let $\tau_R(X)$ be the nano topology on $\mathcal{U}$ with respect to X. Then $[\tau_R(X)]^C$, whose elements are A^C for $A \in \tau_R(X)$, is a topology on $\mathcal{U}$.

Remark 3.12 : $[\tau_N(F)]^C$ is called the dual neutrosophic nano topology of $\tau_N(F)$. Elements of $[\tau_N(F)]^C$ are called neutrosophic nano closed sets. Thus, we note that a neutrosophic set N(G) of $\mathcal{U}$ is neutrosophic nano closed in $\tau_N(F)$ if and only if $\mathcal{U} - N(G)$ is neutrosophic nano open in $\tau_N(F)$.

Example 3.13 : Let $\mathcal{U} = \{p_1, p_2, p_3\}$ be the universe of discourse. Let $\mathcal{U}/R = \{\{p_1, p_2\}, \{p_3\}\}$ be an equivalence relation on $\mathcal{U}$ and $A = \{< p_1, (0.7, 0.6, 0.5) >, < p_2, (0.3, 0.4, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}$ be a neutrosophic set on $\mathcal{U}$ then $\underline{N}(A) = \{< p_1, (0.3, 0.4, 0.5) >, < p_2, (0.3, 0.4, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}$, $\overline{N}(A) = \{< p_1, (0.7, 0.6, 0.5) >, < p_2, (0.7, 0.6, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}$, $B(A) = \{< p_1, (0.5, 0.6, 0.5) >, < p_2, (0.5, 0.6, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}$. Then the collection $\tau_N(A) = \{0_N, 1_N, \{< p_1, (0.3, 0.4, 0.5) >, < p_2, (0.3, 0.4, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}, \{< p_1, (0.7, 0.6, 0.5) >, < p_2, (0.7, 0.6, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}, \{< p_1, (0.5, 0.6, 0.5) >, < p_2, (0.5, 0.6, 0.5) >, < p_3, (0.1, 0.5, 0.1) >\}\}$ is a neutrosophic nano topology on $\mathcal{U}$ and $[\tau_N(A)]^C$ is also a neutrosophic nano topology on $\mathcal{U}$. Thus $\tau_{\mathcal{I}}(A) = \{0_N, 1_N, \{< p_1, (0.3, 0.5) >, < p_2, (0.3, 0.5) >, < p_3, (0.1, 0.1) >\}, \{< p_1, (0.7, 0.5) >, < p_2, (0.7, 0.5) >, < p_3, (0.1, 0.1) >\}, \{< p_1, (0.5, 0.5) >, < p_2, (0.5, 0.5) >, < p_3, (0.1, 0.1) >\}\}$ and $\tau_{\mathcal{F}}(A) = \{0_N, 1_N, \{< p_1, (0.3) >, < p_2, (0.3) >, < p_3, (0.1) >\}, \{< p_1, (0.7) >, < p_2, (0.7) >, < p_3, (0.1) >\}, \{< p_1, (0.5) >, < p_2, (0.5) >, < p_3, (0.1) >\}\}$ are the intuitionistic nano topology and fuzzy nano topology.

Remark 3.14 : In neutrosophic nano topological space, the neutrosophic nano boundary cannot be empty. Since the difference between neutrosophic nano upper and neutrosophic nano lower approximations is defined here as the maximum and minimum of the values in the neutrosophic sets.

Proposition 3.15 : Let $\mathcal{U}$ be a non-empty finite universe and F be a neutrosophic set on $\mathcal{U}$. Then the following statements hold:

(i) The collection $\tau_N(F) = \{0_N, 1_N\}$, is the indiscrete neutrosophic nano topology on $\mathcal{U}$.

(ii) If $\underline{N}(F) = \overline{N}(F) = N(F)$, then the neutrosophic nano topology, $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), BN(F)\}$.

(iii) If $\underline{N}(F) = BN(F)$, then $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F)\}$ is a neutrosophic nano topology

(iv) If $\overline{N}(F) = BN(F)$ then $\tau_N(F) = \{0_N, 1_N, \overline{N}(F), BN(F)\}$.

(v) The collection $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), BN(F)\}$ is the discrete neutrosophic nano topology on $\mathcal{U}$.

4 Neutrosophic nano closure and interior

In this section we have defined neutrosophic nano closure and neutrosophic nano interior on neutrosophic nano topological space. Based on this we also prove some properties.

Definition 4.1 : If $(\mathcal{U}, \tau_N(F))$ is a neutrosophic nano topological space with respect to neutrosophic subset of $\mathcal{U}$ and if A be any neutrosophic subset of $\mathcal{U}$, then the neutrosophic nano interior of A is defined as the union of all neutrosophic nano open subsets of A and it is denoted by $N_{\mathcal{F}}int(A)$. That is, $N_{\mathcal{F}}int(A)$ is the largest neutrosophic nano open subset of A. The neutrosophic nano closure of A is defined as the intersection of all neutrosophic nano closed sets containing A and it is denoted by $N_{\mathcal{F}}cl(A)$. That is, $N_{\mathcal{F}}cl(A)$ is the smallest neutrosophic nano closed set containing A.

Remark 4.2 : Let $(\mathcal{U}, \tau_N(F))$ be a neutrosophic nano topological space with respect to F where F is a neutrosophic subset of $\mathcal{U}$. The neutrosophic nano closed sets in $\mathcal{U}$ are 0_N,1_N, $(\underline{N}(F))^C$, $(\overline{N}(F))^C$ and $(B_N(F))^C$.

Theorem 4.3 *[8]*: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space with respect to $X \subseteq \mathcal{U}$ then $\mathcal{N}cl(X) = \mathcal{U}$.

Remark 4.4 : The above theorem need not be true for all neutrosophic nano topological space $(\mathcal{U}, \tau_N(F))$ with respect to F where F is a neutrosophic subset of $\mathcal{U}$. That is $N_{\mathcal{F}}cl(A)$ *need not be equal to* $\mathcal{U}$ which can be shown by the following example.

Example 4.5 : Let $\mathcal{U} = \{p_1, p_2, p_3, p_4, p_5\}$ be the universe of discourse. Let $\mathcal{U}/R = \{\{p_1, p_4\}, \{p_2, p_3\}, \{p_5\}\}$ be an equivalence relation on $\mathcal{U}$ and $A = \{< p_1, (0.2, 0.3, 0.4) > , < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$ be a neutrosophic set on $\mathcal{U}$. Then $\underline{N}(A) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$, $\overline{N}(A) = \{< p_1, (0.2, 0.3, 0.4) > , < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$ $B(A) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) > , < p_5, (0.2, 0.4, 0.4) >\}$. Now we have $\tau_N(A) = \{0_N, 1_N, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) > , < p_5, (0.2, 0.4, 0.4) >\}\}$ which is a neutrosophic nano topology on $\mathcal{U}$. $[\tau_N(A)]^c = \{0_N, 1_N, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}, \{< p_1, (0.2, 0.3, 0.4) > , < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$. Here $N_{\mathcal{F}}cl(A) \neq \mathcal{U}$

Theorem 4.6 : Let $(\mathcal{U}, \tau_N(F))$ be a neutrosophic nano topological space with respect to F where F is a neutrosophic subset of $\mathcal{U}$. Let A and B be neutrosophic subsets of $\mathcal{U}$. Then the following statements hold:

(i) $A \subseteq N_{\mathcal{F}}cl(A)$.

(ii) A is nano closed if and only if $N_{\mathcal{F}}cl(A) = A$.

(iii) $N_{\mathcal{F}}cl(0_N) = 0_N$ and $N_{\mathcal{F}}cl(1_N) = 1_N$.

(iv) $A \subseteq B \Rightarrow N_{\mathcal{F}}cl(A) \subseteq N_{\mathcal{F}}cl(B)$.

(v) $N_{\mathcal{F}}cl(A \cup B) = N_{\mathcal{F}}cl(A) \cup N_{\mathcal{F}}cl(B)$.

(vi) $N_{\mathcal{F}}cl(A \cap B) \subseteq N_{\mathcal{F}}cl(A) \cap N_{\mathcal{F}}cl(B)$.

(vii) $N_{\mathcal{F}}cl(N_{\mathcal{F}}cl(A)) = N_{\mathcal{F}}cl(A)$.

Proof:

(i) By definition of neutrosophic nano closure, $A \subseteq N_{\mathcal{F}}cl(A)$.

(ii) If A is neutrosophic nano closed, then A is the smallest neutrosophic nano closed set containing itself and hence $N_{\mathcal{F}}$cl(A) = A. Conversely, if $N_{\mathcal{F}}$cl(A) = A, then A is the smallest neutrosophic nano closed set containing itself and hence A is neutrosophic nano closed.

(iii) Since 0_N and 1_N are neutrosophic nano closed in $(\mathcal{U}, \tau_N(F))$, $N_{\mathcal{F}}cl(0_N) = 0_N$ and $N_{\mathcal{F}}cl(1_N) = 1_N$.

(iv) If $A \subseteq B$, since $B \subseteq N_{\mathcal{F}}cl(B)$, then $A \subseteq N_{\mathcal{F}}cl(B)$. That is, $N_{\mathcal{F}}$cl(B) is a Neutrosophic nano closed set containing A. But $N_{\mathcal{F}}$cl(A) is the smallest Neutrosophic nano closed set containing A. Therefore, $N_{\mathcal{F}}cl(A) \subseteq N_{\mathcal{F}}cl(B)$.

(v) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $N_{\mathcal{F}}cl(A) \subseteq N_{\mathcal{F}}cl(A \cup B)$ and $N_{\mathcal{F}}cl(B) \subseteq N_{\mathcal{F}}cl(A \cup B)$. Therefore, $N_{\mathcal{F}}cl(A) \cup N_{\mathcal{F}}cl(B) \subseteq N_{\mathcal{F}}cl(A \cup B)$. By the fact that $A \cup B \subseteq N_{\mathcal{F}}cl(A) \cup N_{\mathcal{F}}cl(B)$, and since $N_{\mathcal{F}}cl(A \cup B)$ is the smallest nano closed set containing $A \cup B, so N_{\mathcal{F}}cl(A \cup B) \subseteq N_{\mathcal{F}}cl(A) \cup N_{\mathcal{F}}cl(B)$. Thus, $N_{\mathcal{F}}cl(A \cup B) = N_{\mathcal{F}}cl(A) \cup N_{\mathcal{F}}cl(B)$.

(vi) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B, N_{\mathcal{F}}cl(A \cap B) \subseteq N_{\mathcal{F}}cl(A) \cap N_{\mathcal{F}}cl(B)$.

(vii) Since $N_{\mathcal{F}}cl(A)$ is nano closed, $N_{\mathcal{F}}cl(N_{\mathcal{F}}cl(A)) = N_{\mathcal{F}}cl(A)$.

Theorem 4.7 : $(\mathcal{U}, \tau_N(F))$ be a neutrosophic nano topological space with respect to F where F is a neutrosophic subset of $\mathcal{U}$. Let A be a neutrosophic subset of $\mathcal{U}$. Then

(i) $1_N - N_{\mathcal{F}}Int(A) = N_{\mathcal{F}}cl(1_N - A)$.

(ii) $1_N - N_{\mathcal{F}}cl(A) = N_{\mathcal{F}}Int(1_N - A)$.

Remark 4.8 : Taking complements on either side of(i) and (ii) Theorem 4.8, we get $(N_{\mathcal{F}}Int(A)) = 1_N - N_{\mathcal{F}}cl(1_N - A))$ and $(N_{\mathcal{F}}cl(A)) = 1_N - (N_{\mathcal{F}}Int(1_N - A))$.

Example 4.9 : Let $\mathcal{U} = \{a, b, c\}$ and $\mathcal{U}/R = \{\{a, b\}, \{c\}\}$. Let $F = \{< a, (0.4, 0.5, 0.5) > , < b, (0.4, 0.5, 0.5) >, < c, (0.5, 0.5, 0.5) >\}$ be a neutrosophic set on $\mathcal{U}$ then the $\tau_N(A) = \{0_N, 1_N, \{< a, (0.4, 0.5, 0.5) >, < b, (0.4, 0.5, 0.5) >, < c, (0.5, 0.5, 0.5) >\}\}$ is a neutrosophic nano topology on $\mathcal{U}$. $[\tau_N(A)]^c = \{0_N, 1_N, \{< a, (0.5, 0.5, 0.4) >, < b, (0.5, 0.5, 0.4) > , < c, (0.5, 0.5, 0.5) >\}\}$. If $A = \{< a, (0.7, 0.6, 0.5) >, < b, (0.3, 0.4, 0.5) >, < c, (0.7, 0.5, 0.5) > \}$, then $(N_{\mathcal{F}}Int(A))^C = 1_N$ $N_{\mathcal{F}}cl(1_N - A) = 1_N$. That is, $1_N - N_{\mathcal{F}}Int(A) = N_{\mathcal{F}}cl(1_N - A)$ Also, $1_N - N_{\mathcal{F}}cl(A) = \mathcal{N}_{\mathcal{F}}Int(1_N - A) = 0_N$

Theorem 4.10 : Let $(\mathcal{U}, \tau_N(F))$ be a neutrosophic nano topological space with respect to F where F is a neutrosophic subset of $\mathcal{U}$. Let A and B be neutrosophic subsets of $\mathcal{U}$, then the following statements hold:

(i) A is neutrosophic nano open if and only if $N_{\mathcal{F}}Int(A) = A$.

(iii) $N_{\mathcal{F}}Int(0_N) = 0_N$ and $N_{\mathcal{F}}Int(1_N) = 1_N$.

(iv) $A \subseteq B \Rightarrow N_{\mathcal{F}}Int(A) \subseteq N_{\mathcal{F}}Int(B)$.

(v) $N_{\mathcal{F}}Int(A) \cup N_{\mathcal{F}}Int(B) \subseteq N_{\mathcal{F}}Int(A \cup B)$.

(vi) $N_{\mathcal{F}}Int(A \cap B) = N_{\mathcal{F}}Int(A) \cap N_{\mathcal{F}}Int(B)$.

(vii) $N_{\mathcal{F}}Int(N_{\mathcal{F}}Int(A)) = N_{\mathcal{F}}Int(A)$.

Proof:

(i) A is neutrosophic nano open if and only if $1_N - A$ is neutrosophic nano closed, if and only if $N_{\mathcal{F}}cl(1_N - A) = 1_N - A$, if and only if $1_N - N_{\mathcal{F}}cl(1_N - A) = A$ if and only if $N_{\mathcal{F}}Int(A) = A$, by Remark 4.8.

(ii) Since 0_N and 1_N are neutrosophic nano open, $N_{\mathcal{F}}Int(0_N) = 0_N$ and $N_{\mathcal{F}}Int(1_N) = 1_N$.

(iii) $A \subseteq B \Rightarrow 1_N - B \subseteq 1_N - A$. Therefore, $N_{\mathcal{F}}cl(1_N - B) \subseteq N_{\mathcal{F}}cl(1_N - A)$. That is, $1_N - N_{\mathcal{F}}cl(1_N - A) \subseteq 1_N - N_{\mathcal{F}}cl(1_N - B)$. That is, $N_{\mathcal{F}}IntA \subseteq N_{\mathcal{F}}IntB$.

Proof of (iv), (v) and (vi) follow similarly from Theorem 4.7 and Remark 4.8.

Conclusion: Neutrosophic set is a general formal framework, which generalizes the concept of classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, and interval intuitionistic fuzzy set. Since the world is full of indeterminacy, the neutrosophic nano topology found its place into contemporary research world. This paper can be further developed into several possible such as Geographical Information Systems (GIS) field including remote sensing, object reconstruction from airborne laser scanner, real time tracking, routing applications and modeling cognitive agents. In GIS there is a need to model spatial regions with indeterminate boundary and under indeterminacy. Hence this neutrosophic nano topological spaces can also be extended to a neutrosophic spatial region.

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NC-VIKOR Based MAGDM Strategy under Neutrosophic Cubic Set Environment

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Abstract. Neutrosophic cubic set consists of interval neutrosophic set and single valued neutrosophic set simultaneously. Due to its unique structure, neutrosophic cubic set can express hybrid information consisting of single valued neutrosophic information and interval neutrosophic information simultaneously. VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) strategy is an important decision making strategy which selects the optimal alternative by utilizing maximum group utility and minimum of an individual regret. In this paper, we propose VIKOR strategy in neutrosophic cubic set environment, namely NC-VIKOR. We first define NC-VIKOR strategy in neutrosophic cubic set environment to handle multi-attribute group decision making (MAGDM) problems, which means we combine the VIKOR with neutrosophic cubic number to deal with multi-attribute group decision making problems. We have proposed a new strategy for solving MAGDM problems. Finally, we solve MAGDM problem using our newly proposed NC-VIKOR strategy to show the feasibility, applicability and effectiveness of the proposed strategy. Further, we present sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Keywords: MAGDM, NCS, NC-VIKOR strategy.

1. Introduction

Smarandache [1] introduced neutrosophic set (NS) by defining the truth membership function, indeterminacy function and falsity membership function as independent components by extending fuzzy set [2] and intuitionistic fuzzy set [3]. Each of three independent component of NS belons to $[^{-}0, 1^{+}]$. Wang et al. [4] introduced single valued neutrosophic set (SVNS) where each of truth, indeterminacy and falsity membership degree belongs to [0, 1]. Many researchers developed and applied the NS and SVNS in various areas of research such as conflict resolution [5], clustering analysis [6-9], decision making [10-39], educational problem [40, 41], image processing [42-45], medical diagnosis [46, 47], social problem [48, 49]. Wang et al. [50] proposed interval neutrosophic set (INS). Ye [51] defined similarity measure of two interval neutrosophic sets and applied it to solve multi criteria decision making (MCDM) problem. By combining SVNS and INS Jun et al. [52], and Ali et al. [53] proposed neutrosophic cubic set (NCS). Thereafter, Zhan et al. [54] presented two weighted average operators on NCSs and applied the operators for MADM problem. Banerjee et al. [55] introduced the grey relational analysis based MADM strategy in NCS environment. Lu and Ye [56] proposed three cosine measures between NCSs and presented MADM strategy in NCS environment. Pramanik et al. [57] defined similarity measure for NCSs and proved its basic properties and presented a new multi criteria group decision making strategy with linguistic variables in NCS environment. Pramanik et al. [58] proposed the score and accuracy functions for NCSs and prove their basic properties. In the same study, Pramanik et al. [59] developed a strategy for ranking of neutrosophic cubic numbers (NCNs) based on the score and accuracy functions. In the same study, Pramanik et al. [58] first developed a TODIM (Tomada de decisao interativa e multicritévio), called the NC-TODIM and presented new NC-TODIM [58] strategy for solving (MAGDM) in NCS environment. Shi and Ye [59] introduced Dombi aggregation operators of NCSs and applied them for MADM problem. Pramanik et al. [60] proposed an ex-

tended technique for order preference by similarity to ideal solution (TOPSIS) strategy in NCS environment for solving MADM problem. Ye [61] present operations and aggregation method of neutrosophic cubic numbers for MADM. Pramanik et al. [62] presented some operations and properties of neutrosophic cubic soft set.

Opricovic [63] proposed the VIKOR strategy for a MAGDM problem with conflicting attributes [64-65]. In 2015, Bausys and Zavadskas [66] extended the VIKOR strategy to INS environment and applied it to solve MCDM problem. Further, Hung et al. [67] proposed VIKOR method for interval neutrosophic MAGDM. Pouresmaeil et al. [68] proposed an MAGDM strategy based on TOPSIS and VIKOR in SVNS environment. Liu and Zhang [69] extended VIKOR method in neutrosophic hesitant fuzzy set environment. Hu et al. [70] proposed interval neutrosophic projection based VIKOR method and applied it for doctor selection. Selvakumari et al. [70] proposed VIKOR Method for decision making problem using octagonal neutrosophic soft matrix.

VIKOR strategy in NCS environment is yet to appear in the literature.

Research gap:

MAGDM strategy based on NC-VIKOR. This study answers the following research questions:

i. Is it possible to extend VIKOR strategy in NCS environment?

ii. Is it possible to develop a new MAGDM strategy based on the proposed NC-VIKOR method in NCS environment?

Motivation:

The above-mentioned analysis [64-69] describes the motivation behind proposing a novel NC-VIKOR method based MAGDM strategy under the NCS environment. This study develops a novel NC-VIKOR -based MAGDM strategy that can deal with multiple decision-makers.

The objectives of the paper are:

i. To extend VIKOR strategy in NCS environment.

ii. To define aggregation operator.

iii. To develop a new MAGDM strategy based on proposed NC-VIKOR in NCS environment.

To fill the research gap, we propose NC-VIKOR strategy, which is capable of dealing with MAGDM problem in NCS environment.

The main contributions of this paper are summarized below:

i. We developed a new NC-VIKOR strategy to deal with MAGDM problems in NCS environment.

ii. We introduce a neutrosophic cubic number aggregation operator and prove its basic properties.

iii. In this paper, we develop a new MAGDM strategy based on proposed NC-VIKOR method under NCS environment to solve MAGDM problems.

iv. In this paper, we solve a MAGDM problem based on proposed NC-VIKOR method.

The remainder of this paper is organized as follows: In the section 2, we review some basic concepts and operations related to NS, SVNS, NCS. In Section 3, we develop a novel MAGDM strategy based on NC-VIKOR to solve the MADGM problems with NCS environment. In Section 4, we solve an illustrative numerical example using the proposed NC-VIKOR in NCS environment. Then in Section 5, we present the sensitivity analysis. The conclusions of the whole paper and further direction of research are given in Section 6.

2. Preliminaries

Definition 1. Neutrosophic set

Let X be a space of points (objects) with a generic element in X denoted by x, i.e. $x \in X$. A neutrosophic set [1] A in X is characterized by truth-membership function $t_A(x)$, indeterminacy-membership function $i_A(x)$ and falsity-membership function $f_A(x)$, where $t_A(x), i_A(x), f_A(x)$ are the functions from X to $]^-0, 1^+[$ i.e. $t_A, i_A, f_A : X \rightarrow]^-0, 1^+[$ that means $t_A(x), i_A(x), f_A(x)$ are the real standard or non-standard subset of $]0, 1^+[$. Neutrosophic set can be expressed as $A = \{<x, (t_A(x), i_A(x), f_A(x))>: \forall x \in X\}$ and $^-0 \leq t_A(x)+i_A(x)+f_A(x) \leq 3^+$.

Example 1. Suppose that $X = \{x_1, x_2, x_3, ..., x_n\}$ be the universal set of n points. Let A_1 be any neutrosophic set in X. Then A_1 expressed as $A_1 = \{<x_1, (0.7, 0.4, 0.3)>: x_1 \in X\}$.

Definition 2. Single valued neutrosophic set

Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set [4] B in X is expressed as:

$B = \{< x: (t_B(x), i_B(x), f_B(x))>: x \in X\}$, where $t_B(x), i_B(x), f_B(x) \in [0, 1]$.

For each $x \in X$, $t_B(x), i_B(x), f_B(x) \in [0, 1]$ and $0 \leq t_B(x) + i_B(x) + f_B(x) \leq 3$.

Definition 3. Interval neutrosophic set

An interval neutrosophic set [50] $\tilde{A}$ of a non empty set H is expreesed by truth-membership function $t_{\tilde{A}}(h)$ the indeterminacy membership function $i_{\tilde{A}}(h)$ and falsity membership function $f_{\tilde{A}}(h)$. For each $h \in H$, $t_{\tilde{A}}(h)$, $i_{\tilde{A}}(h)$, $f_{\tilde{A}}(h) \subseteq [0, 1]$ and $\tilde{A}$ defined as follows:

$\tilde{A} = \{< h, [\, t^{-}_{\tilde{A}}(h), t^{+}_{\tilde{A}}(h)\,], [\, i^{-}_{\tilde{A}}(h), i^{+}_{\tilde{A}}(h)\,], [\, f^{-}_{\tilde{A}}(h), f^{+}_{\tilde{A}}(h)\,]: \forall h \in H\}$. Here, $t^{-}_{\tilde{A}}(h), t^{+}_{\tilde{A}}(h), i^{-}_{\tilde{A}}(h), i^{+}_{\tilde{A}}(h), f^{-}_{\tilde{A}}(h), f^{+}_{\tilde{A}}(h) : H \to\,]^{-}0, 1^{+}[$ and $^{-}0 \le \sup t^{+}_{\tilde{A}}(h) + \sup i^{+}_{\tilde{A}}(h) + \sup f^{+}_{\tilde{A}}(h) \le 3^{+}$.

Here, we consider $t^{-}_{\tilde{A}}(h), t^{+}_{\tilde{A}}(h), i^{-}_{\tilde{A}}(h), i^{+}_{\tilde{A}}(h), f^{-}_{\tilde{A}}(h), f^{+}_{\tilde{A}}(h) : H \to [0, 1]$ for real applications.

Example 2.

Assume that $H = \{h_1, h_2, h_3, \ldots, h_n\}$ be a non-empty set. Let $\tilde{A}_1$ be any interval neutrosophic set. Then $\tilde{A}_1$ expressed as $\tilde{A}_1 = \{< h_1: [0.30, 0.70], [0.20, 0.45], [0.18, 0.39]: h \in H\}$.

Definition 4. Neutrosophic cubic set

A neutrosophic cubic set [52, 53] in a non-empty set H is defined as $N = \{< h, \tilde{A}(h), A(h) >: \forall h \in H\}$, where $\tilde{A}$ and A are the interval neutrosophic set and neutrosophic set in H respectively. Neutrosophic cubic set can be presented as an order pair $N = < \tilde{A}, A >$, then we call it as neutrosophic cubic (NC) number.

Example 3.

Suppose that $H = \{h_1, h_2, h_3, \ldots, h_n\}$ be a non-empty set. Let N_1 be any NC-number. Then N_1 can be expressed as $N_1 = \{< h_1; [0.35, 0.47], [0.20, 0.43], [0.18, 0.42], (0.7, 0.3, 0.5)>: h_1 \in H\}$.

Some operations of NC-numbers: [52, 53]

i. Union of any two NC-numbers

Let $N_1 = < \tilde{A}_1, A_1 >$ and $N_2 = < \tilde{A}_2, A_2 >$ be any two NC-numbers in a non-empty set H. Then the union of N_1 and N_2 denoted by $N_1 \cup N_2$ is defined as follows:

$N_1 \cup N_2 = < \tilde{A}_1(h) \cup \tilde{A}_2(h), A_1(h) \cup A_2(h)\ \forall h \in H >$, where

$\tilde{A}_1(h) \cup \tilde{A}_2(h) = \{< h, [\max\{ t^{-}_{\tilde{A}_1}(h), t^{-}_{\tilde{A}_2}(h)\}, \max\{ t^{+}_{\tilde{A}_1}(h), t^{+}_{\tilde{A}_2}(h)\}], [\min\{ i^{-}_{\tilde{A}_1}(h), i^{-}_{\tilde{A}_2}(h)\}, \min\{ i^{+}_{\tilde{A}_1}(h), i^{+}_{\tilde{A}_2}(h)\}], [\min\{ f^{-}_{\tilde{A}_1}(h), f^{-}_{\tilde{A}_2}(h)\}, \min\{ f^{+}_{\tilde{A}_1}(h), f^{+}_{\tilde{A}_2}(h)\}]>: h \in H\}$ and $A_1(h) \cup A_2(h) = \{< h, \max\{ t_{A_1}(h), t_{A_2}(h)\}, \min\{ i_{A_1}(h), i_{A_2}(h)\}, \min\{ f_{A_1}(h), f_{A_2}(h)\}>: \forall h \in H\}$.

Example 4.

Assume that

$N_1 = <$ [0.39, 0.47], [0.17, 0.43], [0.18, 0.36], (0.6, 0.3, 0.4)> and $N_2 = <$ [0.56, 0.70], [0.27, 0.42], [0.15, 0.26], (0.7, 0.3, 0.6)> be two NC-numbers. Then $N_1 \cup N_2 =$ < [0.56, 0.7], [0.17, 0.42], [0.15, 0.26], (0.7, 0.3, 0.4)>.

ii. Intersection of any two NC-numbers

Intersection of N_1 and N_2 denoted by $N_1 \cap N_2$ is defined as follows:

$N_1 \cap N_2 = < \tilde{A}_1(h) \cap \tilde{A}_2(h), A_1(h) \cap A_2(h)\ \forall h \in H >$, where $\tilde{A}_1(h) \cap \tilde{A}_2(h) = \{< h, [\min\{ t^{-}_{\tilde{A}_1}(h), t^{-}_{\tilde{A}_2}(h)\}, \min\{ t^{+}_{\tilde{A}_1}(h), t^{+}_{\tilde{A}_2}(h)\}], [\max\{ i^{-}_{\tilde{A}_1}(h), i^{-}_{\tilde{A}_2}(h)\}, \max\{ i^{+}_{\tilde{A}_1}(h), i^{+}_{\tilde{A}_2}(h)\}], [\max\{ f^{-}_{\tilde{A}_1}(h), f^{-}_{\tilde{A}_2}(h)\}, \max\{ f^{+}_{\tilde{A}_1}(h), f^{+}_{\tilde{A}_2}(h)\}]>: h \in H\}$ and $A_1(h) \cap A_2(h) = \{< h, \min\{ t_{A_1}(h), t_{A_2}(h)\}, \max\{ i_{A_1}(h), i_{A_2}(h)\}, \max\{ f_{A_1}(h), f_{A_2}(h)\}>: \forall h \in H\}$.

Example 5.

Assume that

$N_1 = <$ [0.45, 0.57], [0.27, 0.33], [0.18, 0.46], (0.7, 0.3, 0.5)> and $N_2 = <$ [0.67, 0.75], [0.22, 0.44], [0.17, 0.21], (0.8, 0.4, 0.4)> be two NC numbers. Then $N_1 \cap N_2 =$ < [0.45, 0.57], [0.22, 0.33], [0.18, 0.46], (0.7, 0.3, 0.4)>.

iii. Compliment of a NC-number

Let $N_1 = < \tilde{A}_1, A_1 >$ be a NCS in H. Then compliment of $N_1 = < \tilde{A}_1, A_1 >$ is denoted by $N_1^c = \{< h, \tilde{A}_1^c(h), A_1^c(h)>: \forall h \in H\}$.

Here, $\tilde{A}_1{}^c = \{< h, [\, t^{+}_{\tilde{A}_1{}^c}(h), t^{-}_{\tilde{A}_1{}^c}(h)], [\, i^{+}_{\tilde{A}_1{}^c}(h), i^{-}_{\tilde{A}_1{}^c}(h)], [\, f^{+}_{\tilde{A}_1{}^c}(h), f^{-}_{\tilde{A}_1{}^c}(h)]>: \forall h \in H\}$, where, $t^{-}_{\tilde{A}_1{}^c}(h) = \{1\} - t^{+}_{\tilde{A}_1}(h)$, $t^{+}_{\tilde{A}_1{}^c}(h) = \{1\} - t^{+}_{\tilde{A}_1}(h)$, $i^{-}_{\tilde{A}_1{}^c}(h) = \{1\} - i^{+}_{\tilde{A}_1}(h)$, $i^{+}_{\tilde{A}_1{}^c}(h) = \{1\} - i^{+}_{\tilde{A}_1}(h)$, $f^{-}_{\tilde{A}_1{}^c}(h) = \{1\} - f^{+}_{\tilde{A}_1}(h)$, $f^{+}_{\tilde{A}_1{}^c}(h)$

= {1} - $f^{+}_{\tilde{A}_1}(h)$, and $t_{A_1^c}(h)$ = {1} - $t_{A_1}(h)$, $i_{A_1^c}(g)$ = {1} - $i_{A_1}(h)$, $f_{A_1^c}(h)$ = = {1} - $f_{A_1}(h)$.

Example 6.

Assume that N_1 be any NC-number in H in the form:

N_1 = < [.45, .57], [.27, .33], [.18, .46], (.7, .3, .5)>.

Then compliment of N_1 is obtained as N_1^c = < [0.18, 0.46], [0.67, 0.73], [0.45, 0.57], (0.5,0.7, 0.7) >.

iv. Containment

Let $N_1 = <\tilde{A}_1, A_1> = \{<h, [t^{-}_{\tilde{A}_1}(h), t^{+}_{\tilde{A}_1}(h)], [i^{-}_{\tilde{A}_1}(h), i^{+}_{\tilde{A}_1}(h)], [f^{-}_{\tilde{A}_1}(h), f^{+}_{\tilde{A}_1}(h)], (t_{A_1}(h), i_{A_1}(h), f_{A_1}(h))>: h \in H\}$ and $N_2 = <\tilde{A}_2, A_2> = \{<h, [t^{-}_{\tilde{A}_2}(h), t^{+}_{\tilde{A}_2}(h)], [i^{-}_{\tilde{A}_2}(h), i^{+}_{\tilde{A}_2}(h)], [f^{-}_{\tilde{A}_2}(h), f^{+}_{\tilde{A}_2}(h)], (t_{A_2}(h), i_{A_2}(h), f_{A_2}(h))>: h \in H\}$

be any two NC-numbers in a non-empty set H,

then, (i) $N_1 \subseteq N_2$ if and only if

$t^{-}_{\tilde{A}_1}(h) \le t^{-}_{\tilde{A}_2}(h)$, $t^{+}_{\tilde{A}_1}(h) \le t^{+}_{\tilde{A}_2}(h)$,

$i^{-}_{\tilde{A}_1}(h) \ge i^{-}_{\tilde{A}_2}(h)$, $i^{+}_{\tilde{A}_1}(h) \ge i^{+}_{\tilde{A}_2}(h)$,

$f^{-}_{\tilde{A}_1}(h) \ge f^{-}_{\tilde{A}_2}(h)$, $f^{+}_{\tilde{A}_1}(h) \ge f^{+}_{\tilde{A}_2}(h)$

and $t_{A_1}(h) \le t_{A_2}(h)$,

$i_{A_1}(h) \ge i_{A_2}(h)$, $f_{A_1}(h) \ge f_{A_2}(h)$ for all $h \in H$.

Definition 7.

Let N_1= < $[a_1, a_2]$, $[b_1, b_2]$, $[c_1, c_2]$, (a, b, c) > and N_2 = < $[d_1, d_2]$, $[e_1, e_2]$, $[f_1, f_2]$, (d, e, f) > be any two NC-numbers, then distance [58] between them is defined by D (N_1, N_2) =

$$\frac{1}{9}[|a_1 - d_1| + |a_2 - d_2| + |b_1 - e_1| + |b_2 - e_2| + |c_1 - f_1| + |c_2 - f_2| + |a - d| + |b - e| + |c - f|] \quad (1)$$

Definition 2.14: Procedure of normalization

In general, benefit type attributes and cost type attributes can exist simultaneously in MAGDM problem. Therefore the decision matrix must be normalized. Let a_{ij} be a NC-numbers to express the rating value of i-th alternative with respect to j-th attribute (Ψ_j). When attribute $\Psi_j \in$ C or $\Psi_j \in$ G (where C and G be the set of cost type attribute and set of benefit type attributes respectively) The normalized values for cost type attribute and benefit type attribute are calculated by using the following expression (2).

$$a^{*}_{ij} = \begin{cases} a_{ij} & \text{if } \Psi_j \in G \\ 1 - a_{ij} & \text{if } \Psi_j \in C \end{cases} \quad (2)$$

Where, a_{ij} is the performance rating of i th alternative for attribute Ψ_j and max a_j is the maximum performance rating among alternatives for attribute Ψ_j.

VIKOR strategy

The VIKOR strategy is an MCDM or multi-criteria decision analysis strategy to deal with multi-criteria optimization problem. This strategy focuses on ranking and selecting the best alternatives from a set of feasible alternatives in the presence of conflicting criteria for a decision problem. The compromise solution [63, 64] reflects a feasible solution that is the closest to the ideal, and a compromise means an agreement established by mutual concessions. The L_p -metric is used to develop the stategy [65]. The VIKOR strategy is developed using the following form of L_p –metric

$$L_{pi} = \left\{ \sum_{j=1}^{n} \left[\left(\Omega_j^{+} - \Omega_{ij} \right) / \left(\Omega_j^{+} - \Omega_j^{-} \right) \right]^p \right\}^{\frac{1}{p}}$$

$1 \le p \le \infty; i = 1,2,3,....,m.$

In the VIKOR strategy, L_{1i} (as S_i) and $L_{\infty i}$, i (as R_i) are utilized to formulate ranking measure. The solution obtained by min S_i reflects the maximum group utility (‘‘majority” rule), and the solution obtained by min R_i indicates the minimum individual regret of the “opponent”.

Suppose that each alternative is evaluated by each criterion function, the compromise ranking is prepated by comparing the measure of closeness to the ideal alternative. The m alternatives are denoted as A_1, A_2, A_3, ..., A_m. For the alternative A_i, the rating of the j th aspect is denoted by Ω_{ij}, i.e. Ω_{ij} is the value of j th criterion function for the alternative A_i; n is the number of criteria.

The compromise ranking algorithm of the VIKOR strategy is presented using the following steps:

Step 1: Determine the best Ω_j^{+} and the worst Ω_j^{-} values of all criterion functions j =1, 2,..., n . If the

j-th function represents a benefit then:

$\Omega_j^+ = \max_i \Omega_{ij}$, $\Omega_j^- = \min_i \Omega_{ij}$

Step 2: Compute the values S_i and R_i ; i = 1, 2,..., m, by these relations:

$$S_i = \sum_{j=1}^{n} w_j \left(\Omega_j^+ - \Omega_{ij}\right) / \left(\Omega_j^+ - \Omega_j^-\right),$$

$$R_i = \max_j w_j \left(\Omega_j^+ - \Omega_{ij}\right) / \left(\Omega_j^+ - \Omega_j^-\right),$$

Here, w_j is the weight of the criterion that expressss its relative importance.

Step 3: Compute the values Q_i: i = 1, 2,..., m, using the following relation:

$$Q_i = \nu\left(S_i - S^-\right)/\left(S^+ - S^-\right) + (1-\nu)\left(R_i - R^-\right)/\left(R^+ - R\right).$$

Here, $S^+ = \max_i S_i$, $S^- = \min_i S_i$

$R^+ = \max_i R_i$, $R^- = \min_i R_i$

Here, v represents ‘‘the decision making mechanism coefficient” (or ‘‘the maximum group utility”). Here we consider v = 0.5 .

Step 4: Preference ranikng order of the the alternatives is done by sorting the values of S, R and Q in decreasing order.

3. VIKOR strategy for solving MAGDM problem in NCS environment

In this section, we propose a MAGDM strategy in NCS environment. Assume that $\Phi = \{\Phi_1, \Phi_2, \Phi_3, ..., \Phi_r\}$ be a set of r alternatives and $\Psi = \{\Psi_1, \Psi_2, \Psi_3, ..., \Psi_s\}$ be a set of s attributes. Assume that $W = \{w_1, w_2, w_3, ..., w_s\}$ be the weight vector of the attributes, where $w_k \geq 0$ and $\sum_{k=1}^{s} w_k = 1$. Assume that $E = \{E_1, E_2, E_3, ..., E_M\}$ be the set of M decision makers and $\zeta = \{\zeta_1, \zeta_2, \zeta_3, ..., \zeta_M\}$ be the set of weight vector of decision makers, where $\zeta_p \geq 0$ and $\sum_{p=1}^{M} \zeta_p = 1$.

The proposed MAGDM strategy consists of the following steps:

Step: 1. Construction of the decision matrix

Let $DM^p = (a_{ij}^p)_{r\times s}$ (p = 1, 2, 3, …, t) be the p-th decision matrix, where information about the alternative Φ_i provided by the decision maker or expert E_p with respect to attribute Ψ_j (j = 1, 2, 3, …, s). The p-th decision matrix denoted by DM^p (See Equation (3)) is constructed as follows:

$$DM^p = \begin{pmatrix} & \Psi_1 & \Psi_2 & \ldots & \Psi_s \\ \Phi_1 & a_{11}^p & a_{12}^p & \ldots & a_{1s}^p \\ \Phi_2 & a_{21}^p & a_{22}^p & & a_{2s}^p \\ . & . & . & \ldots & . \\ \Phi_r & a_{r1}^p & a_{r2}^p & \ldots & a_{rs}^p \end{pmatrix} \quad (3)$$

Here p = 1, 2, 3,…, M; i = 1, 2, 3,…, r; j = 1, 2, 3,…, s.

Step: 2. Normalization of the decision matrix

In decision making situation, cost type attributes and benefit type attributes play an important role to select the best alternative. Cost type attributes and benefit type attributes may exist simultaneously, so the decision matrices need to be normalized. We use Equation (2) for normalizing the cost type attributes and benefit type attributes. After normalization, the normalized decision matrix (Equation (3)) is represented as follows (see Equation 4):

$$DM^p = \begin{pmatrix} & \Psi_1 & \Psi_2 & \ldots & \Psi_s \\ \Phi_1 & a^{*p}_{11} & a^{*p}_{12} & \ldots & a^{*p}_{1s} \\ \Phi_2 & a^{*p}_{21} & a^{*p}_{22} & & a^{*p}_{2s} \\ . & . & . & \ldots & . \\ \Phi_r & a^{*p}_{r1} & a^{*p}_{r2} & \ldots & a^{*p}_{rs} \end{pmatrix} \quad (4)$$

Here, p = 1, 2, 3,…, M; i = 1, 2, 3,…, r; j = 1, 2, 3,…, s.

Step: 3. Aggregated decision matrix

For obtaining group decision, we aggregate all the individual decision matrices ($DM^p, p = 1, 2, ..., M$) to an aggregated decision matrix (DM) using the neutrosophic cubic numbers weighted aggregation (NCNWA) operator as follows:

$$a_{ij} = \text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) =$$

$$(\zeta_1 a_{ij}^1 \oplus \zeta_2 a_{ij}^2 \oplus \zeta_3 a_{ij}^3 \oplus \ldots \oplus \zeta_M a_{ij}^M) =$$

$$< \Big([\sum_{p=1}^{M} \zeta_p t_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p t_{ij}^{+(p)}], [\sum_{p=1}^{M} \zeta_p i_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p i_{ij}^{+(p)}],$$

$$[\sum_{p=1}^{M} \zeta_p f_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p f_{ij}^{+(p)}], (\sum_{p=1}^{M} \zeta_p t_{ij}^{(p)}, \sum_{p=1}^{M} \zeta_p i_{ij}^{(p)}, \sum_{p=1}^{M} \zeta_p f_{ij}^{(p)}] \Big) > \quad (5)$$

The NCNWA operator satisfies the following properties:

1. Idempotency
2. Monotoncity
3. Boundedness

Property: 1. Idempotency

If all $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M = a$ are equal, then

$$a_{ij} = \text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) = a$$

Proof:

Since $a_{ij}^1 = a_{ij}^2 = \ldots = a_{ij}^M = a$, based on the Equation (5), we get

$$a_{ij} = \text{NCNWA}_\zeta(a_{ij}^1\ a_{ij}^2 \ldots\ a_{ij}^M) =$$
$$(\zeta_1 a_{ij}^1 \oplus \zeta_2 a_{ij}^2 \oplus \zeta_3 a_{ij}^3 \oplus \ldots \oplus \zeta_M a_{ij}^M) =$$
$$(\zeta_1 a \oplus \zeta_2 a \oplus \zeta_3 a \oplus \ldots \oplus \zeta_M a) =$$
$$< \Big([t^- \sum_{p=1}^M \zeta_p, t^+ \sum_{p=1}^M \zeta_p], [i^- \sum_{p=1}^M \zeta_p, i^+ \sum_{p=1}^M \zeta_p],$$
$$[f^- \sum_{p=1}^M \zeta_p, f^+ \sum_{p=1}^M \zeta_p], (t \sum_{p=1}^M \zeta_p, i \sum_{p=1}^M \zeta_p, f \sum_{p=1}^M \zeta_p] \Big) >$$
$$= < \Big([t^-, t^+], [i^-, i^+], [f^-, f^+], (t, i, f] \Big) > = a.$$

Property: 3. Monotonicity

Assume that $\{a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M\}$ and $\{a_{ij}^{*1}, a_{ij}^{*2}, \ldots, a_{ij}^{*M}\}$ be any two set of collections of M NC-numbers with the condition $a_{ij}^p \le a_{ij}^{*p}$ (p = 1, 2, ..., M), then

$$\text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) \le \text{NCNWA}_\zeta(a_{ij}^{*1}, a_{ij}^{*2}, \ldots, a_{ij}^{*M}).$$

Proof:

From the given condition $t_{ij}^{-(p)} \le t_{ij}^{-*(p)}$, we have

$$\zeta_p t_{ij}^{-(p)} \le \zeta_p t_{ij}^{-*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p t_{ij}^{-(p)} \le \sum_{p=1}^M \zeta_p t_{ij}^{-*(p)}.$$

From the given condition $t_{ij}^{+(p)} \le t_{ij}^{+*(p)}$, we have

$$\zeta_p t_{ij}^{+(p)} \le \zeta_p t_{ij}^{+*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p t_{ij}^{+(p)} \le \sum_{p=1}^M \zeta_p t_{ij}^{+*(p)}.$$

From the given condition $i_{ij}^{-(p)} \ge i_{ij}^{-*(p)}$, we have

$$\zeta_p i_{ij}^{-(p)} \ge \zeta_p i_{ij}^{-*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p i_{ij}^{-(p)} \ge \sum_{p=1}^M \zeta_p i_{ij}^{-*(p)}.$$

From the given condition $i_{ij}^{+(p)} \ge i_{ij}^{+*(p)}$, we have

$$\zeta_p i_{ij}^{+(p)} \ge \zeta_p i_{ij}^{+*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p i_{ij}^{+(p)} \ge \sum_{p=1}^M \zeta_p i_{ij}^{+*(p)}.$$

From the given condition $f_{ij}^{-(p)} \ge f_{ij}^{-*(p)}$, we have

$$\zeta_p f_{ij}^{-(p)} \ge \zeta_p f_{ij}^{-*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p f_{ij}^{-(p)} \ge \sum_{p=1}^M \zeta_p f_{ij}^{-*(p)}.$$

From the given condition $f_{ij}^{+(p)} \ge f_{ij}^{+*(p)}$, we have

$$\zeta_p f_{ij}^{+(p)} \ge \zeta_p f_{ij}^{+*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p f_{ij}^{+(p)} \ge \sum_{p=1}^M \zeta_p f_{ij}^{+*(p)}.$$

From the given condition $t_{ij}^{(p)} \le t_{ij}^{*(p)}$, we have

$$\zeta_p t_{ij}^{(p)} \le \zeta_p t_{ij}^{*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p t_{ij}^{(p)} \le \sum_{p=1}^M \zeta_p t_{ij}^{*(p)}.$$

From the given condition $i_{ij}^{(p)} \ge i_{ij}^{*(p)}$, we have

$$\zeta_p i_{ij}^{(p)} \ge \zeta_p i_{ij}^{*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p i_{ij}^{(p)} \le \sum_{p=1}^M \zeta_p i_{ij}^{*(p)}.$$

From the given condition $t_{ij}^{(p)} \le t_{ij}^{*(p)}$, we have

$$\zeta_p f_{ij}^{(p)} \ge \zeta_p f_{ij}^{*(p)}$$
$$\Rightarrow \sum_{p=1}^M \zeta_p f_{ij}^{(p)} \le \sum_{p=1}^M \zeta_p f_{ij}^{*(p)}$$

From the above relations, we obtain

$$\text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) \le \text{NCNWA}_\zeta(a_{ij}^{*1}, a_{ij}^{*2}, \ldots, a_{ij}^{*M}).$$

Property: 2. Boundedness

Let $\{a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M\}$ be any collection of M NC-numbers. If

$$a^+ = < [\max_p\{t_{ij}^{-(p)}\}, [\max_p\{t_{ij}^{+(p)}\}], [\min_p\{i_{ij}^{-(p)}\}, \min_p\{i_{ij}^{+(p)}\}],$$
$$[\min_p\{f_{ij}^{-(p)}\}, \min_p\{f_{ij}^{+(p)}\}], (\max_p\{t_{ij}^p\}, \min_p\{i_{ij}^p\}, \min_p\{f_{ij}^p\}) >$$
$$a^- = < [\min_p\{t_{ij}^{-(p)}\}, [\min_p\{t_{ij}^{+(p)}\}], [\max_p\{i_{ij}^{-(p)}\}, \max_p\{i_{ij}^{+(p)}\}],$$
$$[\max_p\{f_{ij}^{-(p)}\}, \max_p\{f_{ij}^{+(p)}\}], (\min_p\{t_{ij}^p\}, \max_p\{i_{ij}^p\}, \max_p\{f_{ij}^p\}) >.$$

Then, $a^- \le \text{NCNWA}_\zeta(a_{ij}^1\ a_{ij}^2 \ldots\ a_{ij}^M) \le a^+$.

Proof:

From Property 1 and Property 2, we obtain

$$\text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) \ge \text{NCNWA}_\zeta(a^-, a^-, \ldots, a^-) = a^-$$

and

$$\text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) \le \text{NCNWA}_\zeta(a^+, a^+, \ldots, a^+) = a^+.$$

So, we have

$$a^- \le \text{NCNWA}_\zeta(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^M) \le a^+.$$

Therefore, the aggregated decision matrix is defined as follows:

$$DM = \begin{pmatrix} & \Psi_1 & \Psi_2 & \ldots. \Psi_s \\ \Phi_1 & a_{11} & a_{12} \ldots & a_{1s} \\ \Phi_2 & a_{21} & a_{22} & a_{2s} \\ & \ldots\ldots\ldots & \ldots & \\ \Phi_r & a_{r1} & a_{r2} \ldots & a_{rs} \end{pmatrix} \quad (6)$$

Here, i = 1, 2, 3, …, r; j = 1, 2, 3, …, s; p =1, 2, …., M.

Step: 4. Define the positive ideal solution and negative ideal solution

$$a_{ij}^{+} = <[\max_i t_{ij}^{-}, \max_i t_{ij}^{+}], [\min_i i_{ij}^{-}, \min_i i_{ij}^{+}], [\min_i f_{ij}^{-}, \min_i i_{ij}^{+}], (\max_i t_{ij}, \min_i f_{ij}, \min_i f_{ij}) > \quad (7)$$

$$a_{ij}^{-} = <[\min_i t_{ij}^{-}, \min_i t_{ij}^{+}], [\max_i i_{ij}^{-}, \max_i i_{ij}^{+}], [\max_i f_{ij}^{-}, \max_i i_{ij}^{+}], (\min_i t_{ij}, \max_i f_{ij}, \max_i f_{ij}) > \quad (8)$$

Step: 5. Compute Γ_i and Z_i

Γ_i and Z_i represent the average and worst group scores for the alternative A_i respectively with the relations

$$\Gamma_i = \sum_{j=1}^{s} \frac{w_j \times D(a_{ij}^{+}, a_{ij}^{*})}{D(a_{ij}^{+}, a_{ij}^{-})} \quad (9)$$

$$Z_i = \max_j \left\{ \frac{w_j \times D(a_{ij}^{+}, a_{ij}^{*})}{D(a_{ij}^{+}, a_{ij}^{-})} \right\} \quad (10)$$

Here, w_j is the weight of Ψ_j.

The smaller values of Γ_i and Z_i correspond to the better average and worse group scores for alternative A_i, respectively.

Step: 6. Calculate the values of ϕ_i (i = 1, 2, 3, …, r)

$$\phi_i = \gamma \frac{(\Gamma_i - \Gamma^{-})}{(\Gamma^{+} - \Gamma^{-})} + (1-\gamma) \frac{(Z_i - Z^{-})}{(Z^{+} - Z^{-})} \quad (11)$$

Here, $\Gamma_i^{-} = \min_i \Gamma_i, \Gamma_i^{+} = \max_i \Gamma_i$,

$$Z_i^{-} = \min_i Z_i, Z_i^{+} = \max_i Z_i \quad (12)$$

and γ depicts the decision making mechanism coefficient. If $\gamma > 0.5$, it is for "the maximum group utility"; If $\gamma < 0.5$, it is " the minimum regret"; and it is both if $\gamma = 0.5$.

Step: 7. Rank the priority of alternatives

Rank the alternatives by ϕ_i, Γ_i and Z_i according to the rule of traditional VIKOR strategy. The smaller value reflects the better alternative.

4. Illustrative example

To demonstrate the feasibility, applicability and effectiveness of the proposed strategy, we solve a MAGDM problem adapted from [51]. We assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making board involving of three members (E_1, E_2, E_3) who evaluate the four alternatives to invest money. The alternatives are Car company (Φ_1), Food company (Φ_2), Computer company (Φ_3) and Arms company (Φ_4). Decision makers take decision to evaluate alternatives based on the attributes namely, risk factor (Ψ_1), growth factor (Ψ_2), environment impact (Ψ_3). We consider three criteria as benefit type based on Pramanik et al. [58]. Assume that the weight vector of attributes is $W = (0.36, 0.37, 0.27)^T$ and weight vector of decision makers or experts is $\zeta = (0.26, 0.40, 0.34)^T$. Now, we apply the proposed MAGDM strategy using the following steps.

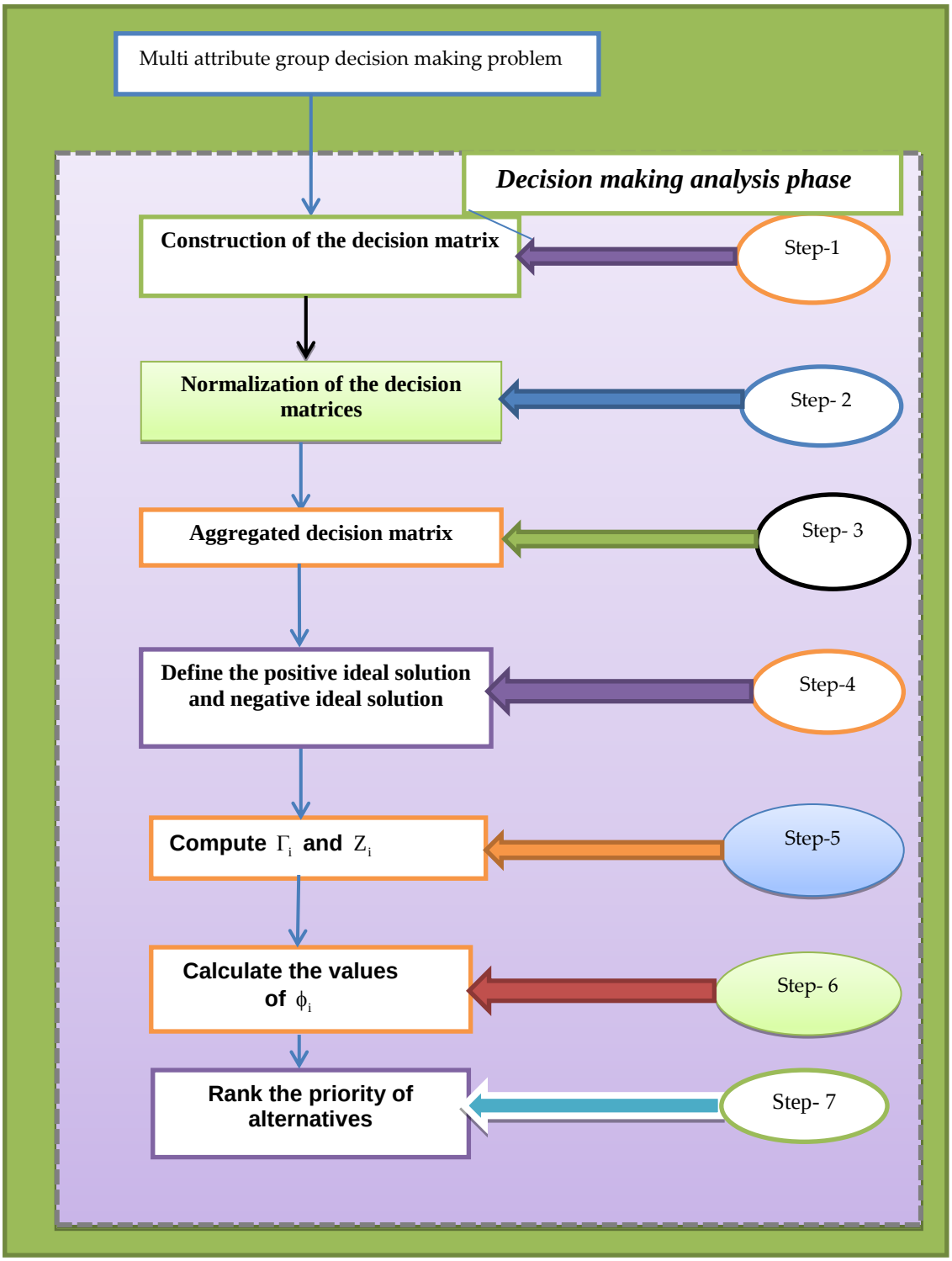

Figure.1 Decision making procedure of proposed MAGDM method

Step: 1. Construction of the decision matrix We construct the decision matrices as follows:

……..

Decision matrix for DM^1 in NCN form

$$\begin{pmatrix} & \Psi_1 & \Psi_2 & \Psi_3 \\ \Phi_1 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \Phi_2 & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \Phi_3 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \Phi_4 & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \end{pmatrix} \quad (13)$$

Decision matrix for DM^2 in NCN form

$$\begin{pmatrix} & \Psi_1 & \Psi_2 & \Psi_3 \\ \Phi_1 & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \Phi_2 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \Phi_3 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \Phi_4 & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \end{pmatrix} \quad (14)$$

Decision matrix for DM^3 in NC-number form

$$\begin{pmatrix} & \Psi_1 & \Psi_2 & \Psi_3 \\ \Phi_1 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \Phi_2 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \Phi_3 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> \\ \Phi_4 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> \end{pmatrix} \quad (15)$$

Step: 2. Normalization of the decision matrix

Since all the criteria are considered as benefit type, we do not need to normalize the decision matrices (DM^1, DM^2, DM^3).

Step: 3. Aggregated decision matrix

Using equation eq. (5), the aggregated decision matrix of (13, 14, 15) is presented below:

$$\begin{pmatrix} & \Psi_1 & \Psi_2 & \Psi_3 \\ \Phi_1 & <[.44,.56],[.36,.46],[.36,.51],(.56,.46,.50)> & <[.48,.60],[.32,.42],[.32,.42],(.60,.42,.42)> & <[.62,.80],[.18,.28],[.18,.28],(.80,.28,.28)> \\ \Phi_2 & <[.45,.58],[.35,.45],[.35,.47],(.58,.45,.47)> & <[.50,.64],[.30,.40],[.30,.40],(.64,.40,.40)> & <[.60,.76],[.20,.30],[.20,.30],(.76,.30,.30)> \\ \Phi_3 & <[.62,.80],[.18,.28],[.18,.28],(.80,.28,.28)> & <[.64,.84],[.16,.26],[.16,.32],(.84,.26,.32)> & <[.47,.60],[.33,.43],[.33,.47],(.60,.43,.47)> \\ \Phi_4 & <[.56,.73],[.24,.34],[.24,.41],(.73,.34,.41)> & <[.40,.50],[.40,.50],[.40,.50],(.50,.50,.50)> & <[.56,.73],[.24,.34],[.24,.37],(.73,.34,.37)> \end{pmatrix} \quad (16)$$

Step: 4. Define the positive ideal solution and negative ideal solution

The positive ideal solution a_{ij}^{+} =

$$\begin{matrix} \Psi_1 & \Psi_2 & \Psi_3 \\ <[.62,.80],[.18,.28],[.18,.28],(.80,.28,.28)> & <[.64,.84],[.16,.26],[.16,.32],(.84,.26,.32)> & <[.62,.80],[.18,.28],[.18,.28],(.80,.28,.28)> \end{matrix}$$

and the negative ideal solution

a_{ij}^{-} =

$$\begin{matrix} \Psi_1 & \Psi_2 & \Psi_3 \\ <[.44,.56],[.36,.46],[.36,.51],(.56,.46,.50)> & <[.40,.50],[.40,.50],[.40,.50],(.50,.50,.50)> & <[.47,.60],[.33,.43],[.33,.43],(.60,.43,.47)> \end{matrix}$$

……

Step: 5. Compute Γ_i and Z_i

Using Equation (9) and Equation (10), we obtain

$$\Gamma_1=\left(\frac{0.36\times 0.2}{0.37}\right)+\left(\frac{0.37\times 0.16}{0.25}\right)+\left(\frac{0.27\times 0}{0.16}\right)=0.43,$$

$$\Gamma_2=\left(\frac{0.36\times 0.18}{0.37}\right)+\left(\frac{0.37\times 0.14}{0.25}\right)+\left(\frac{0.27\times 0.02}{0.16}\right)=0.42,$$

$$\Gamma_3=\left(\frac{0.36\times 0}{0.37}\right)+\left(\frac{0.37\times 0}{0.25}\right)+\left(\frac{0.27\times 0.19}{0.16}\right)=0.32,$$

$$\Gamma_4=\left(\frac{0.36\times 0.08}{0.37}\right)+\left(\frac{0.37\times 0.25}{0.25}\right)+\left(\frac{0.27\times 0.07}{0.16}\right)=0.57.$$

And

$$Z_1=\max\left\{\left(\frac{0.36\times 0.2}{0.37}\right),\left(\frac{0.37\times 0.16}{0.25}\right),\left(\frac{0.27\times 0}{0.16}\right)\right\}=0.24,$$

$$Z_2=\max\left\{\left(\frac{0.36\times 0.18}{0.37}\right),\left(\frac{0.37\times 0.14}{0.25}\right),\left(\frac{0.27\times 0.02}{0.16}\right)\right\}=0.21,$$

$$Z_3=\max\left\{\left(\frac{0.36\times 0}{0.37}\right),\left(\frac{0.37\times 0}{0.25}\right),\left(\frac{0.27\times 0.19}{0.16}\right)\right\}=0.32,$$

$$Z_4=\max\left\{\left(\frac{0.36\times 0.08}{0.37}\right),\left(\frac{0.37\times 0.25}{0.25}\right),\left(\frac{0.27\times 0.07}{0.16}\right)\right\}=0.37.$$

Step: 6. Calculate the values of ϕ_i

Using Equations (11), (12) and $\gamma=0.5$, we obtain

$$\phi_1 = 0.5\times\frac{(0.43-0.32)}{0.25}+0.5\times\frac{(0.24-0.21)}{0.16}=0.31,$$
$$\phi_2 = 0.5\times\frac{(0.42-0.32)}{0.25}+0.5\times\frac{(0.21-0.21)}{0.16}=0.2,$$
$$\phi_3 = 0.5\times\frac{(0.32-0.32)}{0.25}+0.5\times\frac{(0.32-0.21)}{0.16}=0.34,$$
$$\phi_4 = 0.5\times\frac{(0.57-0.32)}{0.25}+0.5\times\frac{(0.37-0.21)}{0.16}=1.$$

Step: 7. Rank the priority of alternatives

The preference order of the alternatives based on the traditional rules of the VIKOR startegy is $\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$.

5. The influence of parameter γ

Table 1 shows how the ranking order of alternatives (Φ_i) changes with the change of the value of γ

Values of γ	Values of ϕ_i	Preference order of alternatives
γ = 0.1	ϕ_1 = 0.22, ϕ_2 = **0.04**, ϕ_3 = 0.62, ϕ_4 = 1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
γ = 0.2	ϕ_1 = 0.24, ϕ_2 = **0.08**, ϕ_3 = 0.55, ϕ_4 = 1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
γ = 0.3	ϕ_1 = 0.26, ϕ_2 = **0.12**, ϕ_3 = 0.48, ϕ_4 = 1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
γ = 0.4	ϕ_1 = 0.29, ϕ_2 = **0.16**, ϕ_3 = 0.41, ϕ_4 = 1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
γ = 0.5	ϕ_1 = 0.31, ϕ_2 = **0.2**, ϕ_3 = 0.34, ϕ_4 = 1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
γ = 0.6	ϕ_1 = 0.34, ϕ_2 = **0.24**, ϕ_3 = 0.28, ϕ_4 = 1	$\Phi_2 \succ \Phi_3 \succ \Phi_1 \succ \Phi_4$
γ = 0.7	ϕ_1 = 0.36, ϕ_2 = 0.28, ϕ_3 = **0.21**, ϕ_4 = 1	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$
γ = 0.8	ϕ_1 = 0.39, ϕ_2 = 0.32, ϕ_3 = **0.14**, ϕ_4 = 1	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$
γ = 0.9	ϕ_1 = 0.42, ϕ_2 = 0.36, ϕ_3 = **0.07**, ϕ_4 = 1	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$

Table1. Values of ϕ_i (i = 1, 2, 3, 4) and ranking of alternatives for different values of γ.

Figure 2 represents the graphical representation of alternatives (A_i) versus ϕ_i (i = 1, 2, 3, 4) for different values of γ.

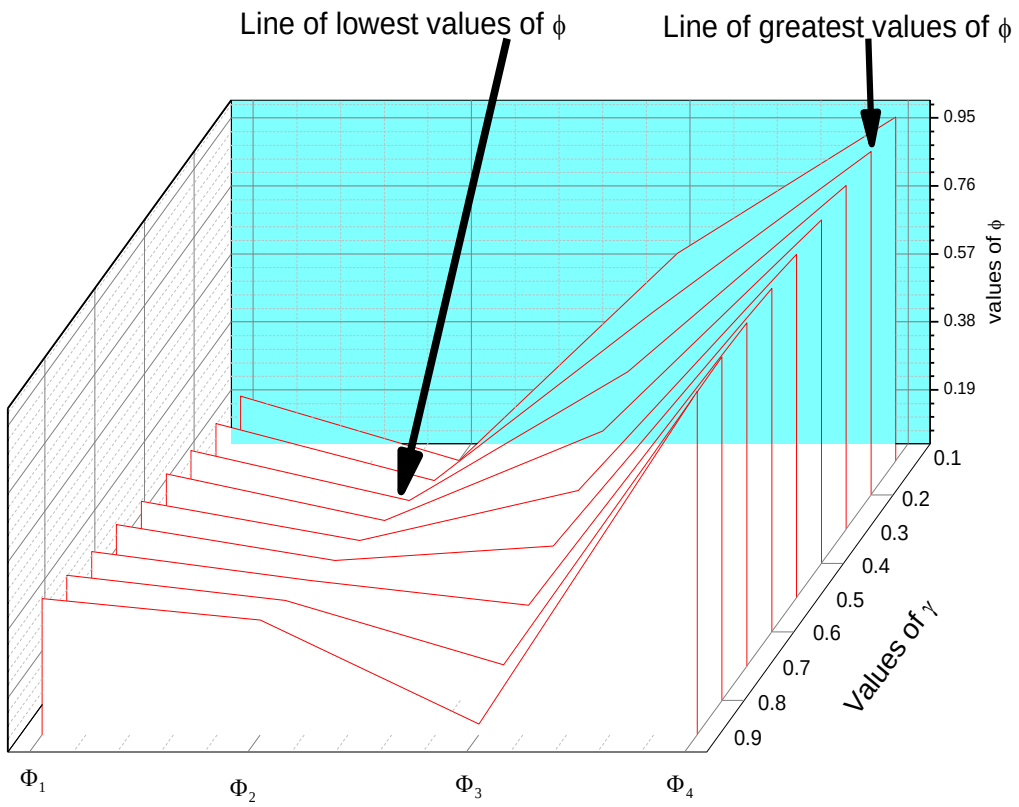

Fig 2. Graphical representation of ranking of alternatives for different values of γ .

6. Conclusions

In this paper, we have extended the traditional VIKOR strategy to NC-VIKOR. We introduced neutrosophic cubic numbers weighted aggregation (NCNWA) operator and applied it to aggregate the individual opinion to group opinion prove its three properties. We develpoed a novel NC-VIKOR based MAGDM strategy in neutrosophic cubic set environment. Finally, we solve a MAGDM problem to show the feasibility, applicability and efficiency of the proposed MAGDM strategy. We present a sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives. The proposed NC-VIKOR based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection [28, 74], teacher selection [75], renewable energy selection[70], fault diagnosis[71], brick selection [76, 77], weaver selection [78], etc.

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Contributions of Selected Indian Researchers to Multi Attribute Decision Making in Neutrosophic Environment: An Overview

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Abstract Multi-attribute decision making (MADM) is a mathematical tool to solve decision problems involving conflicting attributes. With the increasing complexity, uncertainty of objective things and the neutrosophic nature of human thought, more and more attention has been paid to the investigation on multi attribute decision making in neutrosophic environment, and convincing research results have been reported in the literature. While modern algebra and number theory have well documented and established roots deep into India's ancient scholarly history, the understanding of the springing up of neutrosophics, specifically neutrosophic decision making, demands a closer inquiry. The objective of the study is to present a brief review of the pioneering contributions of personalities as diverse as those of P. P. Dey, K. Mondal, P. Biswas, D. Banerjee, S. Dalapati, P. K. Maji, A. Mukherjee, T. K. Roy, B. C. Giri, H. Garg, S. Bhattacharya. A survey of various concepts, issues, etc. related to neutrosophic decision making is discussed. New research direction of neutrosophic decision making is also provided.

Keywords: Bipolar neutrosophic sets, VIKOR method, multi attribute group decision making.

1 Introduction

Every human being has to make decision in every sphere of his/her life. So decision making should be pragmatic and elegant. Decision making involves multi attributes. Multi attribute decision making (MADM) refers to making selections among some courses of actions in the presence of multiple, usually conflicting attributes. MADM is the most well-known branch of decision making. To solve a MADM one needs to employ sorting and ranking (see Figure 1).

It has been widely recognized that most real world decisions take place in uncertain environment where crisp values cannot capture the reflection of the complexity, indeterminacy, inconsistency and uncertainty of the problem.

To deal with crisp MADM problem [1], classical set or crisp set [2] is employed. The classical MADM generally assumes that all the criteria and their respective weights are expressed in terms of crisp numbers and, thus, the rating and the ranking of the alternatives are determined. However, practical decision making problem involves imprecision or vagueness. Imprecision or vagueness may occur from different sources such as unquantifiable information, incomplete information, non-obtainable information, and partial ignorance.

To tackle uncertainty, Zadeh [3] proposed the fuzzy set by introducing membership degree of an element. Different strategies [4-9] have been proposed for dealing with MADM in fuzzy environment. In fuzzy set, non-membership membership function is the complement of membership function. However, non-membership function may be independent in real situation. Sensing this, Atanassov [10] proposed intuitionistic fuzzy set by incorporating non-membership as an independent component. Many MADM strategies [11-14] in intuitionistic fuzzy environment have been studied in the literature. Deschrijver and Kerre [15] proved that intuitionistic fuzzy set is equivalent to interval valued fuzzy set [16], an extension of fuzzy set.

In real world decision making often involves incomplete, indeterminate and inconsistent information. Fuzzy set and intuitionistic fuzzy set

cannot deal with the situation where indeterminacy component is independent of truth and falsity components. To deal with this situation, Smarandache [17] defined neutrosophic set. In 2005, Wang et al. [18] defined interval neutrosophic set. In 2010, Wang et al. [19] introduced the single valued neutrosophic set (SVNS) as a sub class of neutrosophic set. SVNS have caught much attention of the researchers. SVNS have been applied in many areas such as conflict resolution [20], decision making [21-30], image processing [31-33], medical diagnosis [34], social problem [35-36], and so on. In 2013, a new journal, "Neutrosophic Sets and Systems" came into being to propagate neutrosophic study, which can be seen in the journal website, namely, http://fs.gallup.unm.edu/nss. By hybridizing the concept of neutrosophic sets or SVNSs with the various established sets, several neutrosophic hybrid sets have been introduced in the literature such as neutrosophic soft sets [37], neutrosophic soft expert set [38], single valued neutrosophic hesitant fuzzy sets [39], interval neutrosophic hesitant sets [40], interval neutrosophic linguistic sets [41], rough neutrosophic set [42, 43], interval rough neutrosophic set [44], bipolar neutrosophic set [45], bipolar rough neutrosophic set [46], tri-complex rough neutrosophic set [47], hyper complex rough neutrosophic set [48], neutrosophic refined set [49], bipolar neutrosophic refined sets [50], neutrosophic cubic set [51], etc.

So many new areas of decision making in neutrosophic hybrid environment began to emerge. Young researchers demonstrate great interest to conduct research on decision making in neutrosophic as well as neutrosophic hybrid environment. According to Pramanik [52], the concept of neutrosophic set was initially ignored, criticized by many [53, 54], while it was supported only by a very few, mostly young, unknown, and uninfluential researchers. As we see Smarandache [55, 55, 56, 57] leads from the front and makes the paths for research by publishing new books, journal articles, monographs, etc. In India, W. B. V. Kandasamy [58, 59] did many research works on neutrosophic algebra, neutrosophic cognitive maps, etc. She is a well-known researcher in neutrosophic study. Pramanik and Chackrabarti [36] and Pramanik [60, 61] did some work on neutrosophic related problems. Initially, publishing neutrosophic research paper in a recognized journal was a hard work. Pramanik and his colleagues were frustrated by the rejection of several neutrosophic research papers without any valid reasons. After the publication of the International Journal namely, "Neutrosophic Sets and Systems" Pramanik and his colleagues explored the area of decision making in neutrosophic environment to establish their research work.

In 2016, to present history of neutrosophic theory and applications, Smarandache [62] published an edited volume comprising of the short biography and research work of neutrosophic researchers. "*The Encyclopedia of Neutrosophic Researchers*" includes the researchers, who published neutrosophic papers, books, or defended neutrosophic master theses or Ph. D. dissertations. It encourages researchers to conduct study in neutrosophic environment. The fields of neutrosophics have been extended and applied in various fields, such as artificial intelligence, data mining, soft computing, image processing, computational modelling, robotics, medical diagnosis, biomedical engineering, investment problems, economic forecasting, social science, humanistic and practical achievements, and decision making. Decision making in incomplete / indeterminate / inconsistent information systems has been deeply studied by the Indian researchers. New trends in neutrosophic theory and applications can be found in [62-67].

Considering the potentiality of SVNS and its various extensions and their importance of decision making, we feel a sense of commitment to survey the contribution of Indian mathematicians to multi attribute decision making. The venture is exclusively new and therefore it may be considered as an exploratory study.

Research gap:

Survey of new research in MADM conducted by the Indian researchers.

Statement of the problem:

Contributions of selected Indian researchers to multi-attribute decision making in neutrosophic environment: An overview.

Motivation:

The above-mentioned analysis describes the motivation behind the present study.

Objectives of the study

The objective of the study is:

- To present a brief review of the pioneering contributions of personalities as diverse as those

of Dr. Partha Pratim Dey, Dr. Pranab Biswas, Dr. Durga Banerjee, Mr. Kalyan Mondal, Shyamal Dalapati, Dr. P. K. Maji, Prof. T. K. Roy, Prof. B. C. Giri, Prof. Anjan Mukherjee, Dr. Harish Garg and Dr. Sukanto Bhattacharya.

Rest of the paper is organized as follows: In section 2, we review some basic concepts related to neutrosophic set. Section 3 presents the contribution of the selected Indian researchers. Section 4 presents conclusion and future scope of research.

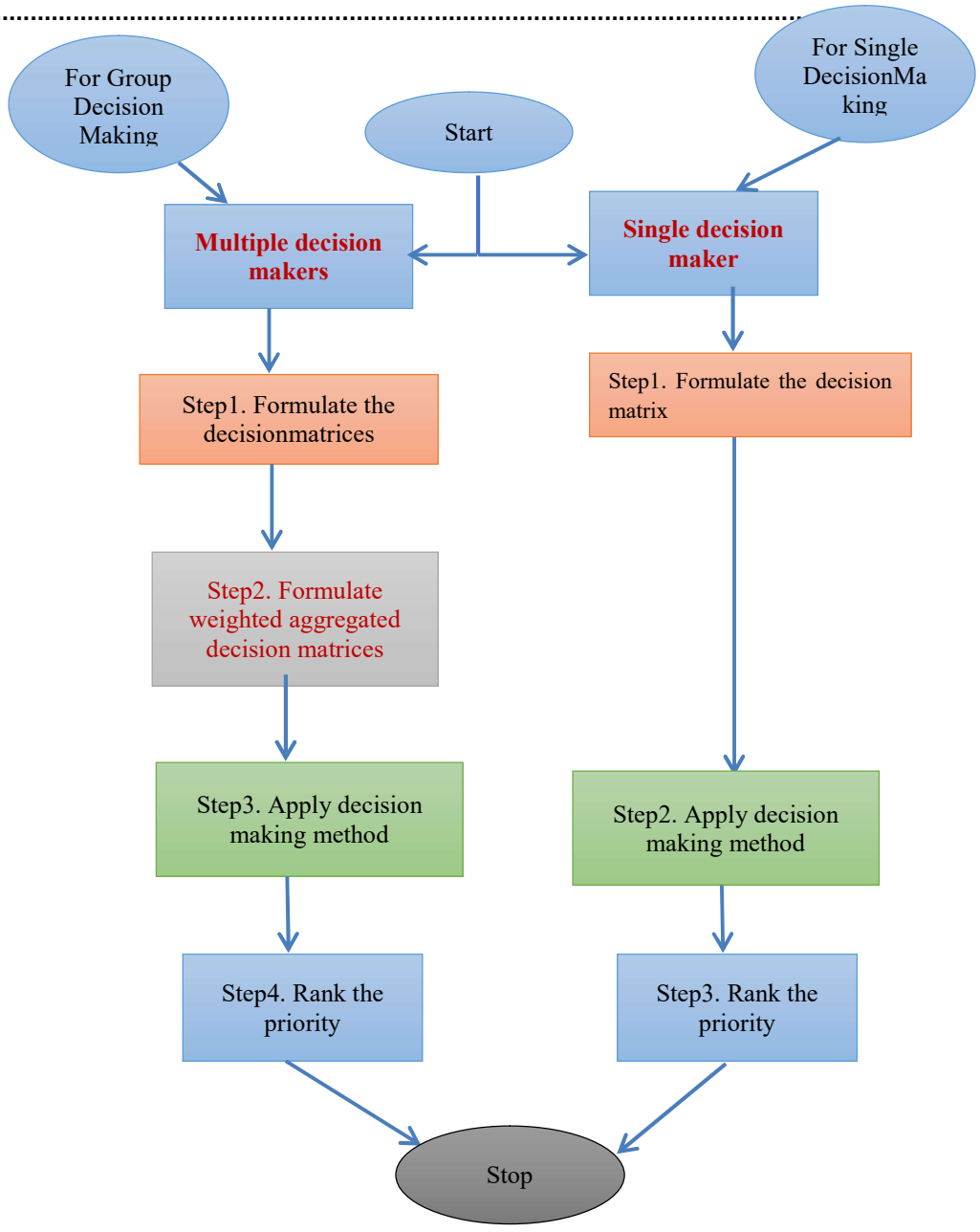


Figure 1. Decision making steps

2. Preliminaries

In this section we recall some basic definitions related to this topic.

Definition.2.1 Neutrosophic Set

Let X be the universe. A neutrosophic set (NS) [17] P in X is characterized by a truth membership function T_P, an indeterminacy membership function I_P and a falsity membership function F_P where T_P, I_P and F_P are real standardor non-standard subset of $]^-0,1^+[$. It can be defined as:

$P=\{<x,(T_P(x),I_P(x),F_P(x))>: x\epsilon X, T_P,I_P,F_P \epsilon]^-0,1^+[\}$

There is no restriction on the sum of $T_P(x),I_P(x)$ and $F_P(x)$ and so $0^- \leq T_P(x)+I_P(x)+F_P(x) \leq 3^+$.

Definition 2.2 Single valued neutrosophic set

Let X be a space of points (objects) with generic element in X denoted by x. A single valued neutrosophic set [19] P is characterized by a truth-membership function $T_P(x)$, an indeterminacy-membership function $I_P(x)$, and a falsity-membership function $F_P(x)$. For each point x in X, $T_P(x),I_P(x),F_P(x) \in [0, 1]$. A SVNS A can be written as:

$A = \{<x:T_P(x),I_P(x),F_P(x)>, x \in X\}$.

Definition 2.3 Interval valued neutrosophic set

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set [18] P is characterized by an interval truth-membership function $T_P(x)=[T_P^L, T_P^U]$, an interval indeterminacy-membership function $I_P(x)=[I_P^L, I_P^U]$, and an interval falsity-membership function $F_P(x)=[F_P^L, F_P^U]$. For each point $x\epsilon X$, $T_P(x)$, $I_P(x)$, $F_P(x) \subset [0, 1]$. An IVNS P can be written as:

$P = \{< x: T_P(x),I_P(x),F_P(x)>x \in X\}$.

Definition 2.4: Bipolar neutrosophic set

A bipolar neutrosophic set [45] P in X is defined as an object of the form $P=\{<x, T^m(x),I^m(x),F^m(x), T^n(x),I^n(x), F^n(x)>: x \in X\}$, where $T^m, I^m, F^m: X \rightarrow [1, 0]$ and $T^n, I^n, F^n: X \rightarrow [-1, 0]$. The positive membership degree $T^m(x)$, $I^m(x)$, $F^m(x)$ denotes respectively the truth membership, indeterminate membership and false membership degree of an element $\in X$ corresponding to a bipolar neutrosophic set P and the negative membership degree $T^n(x)$, $I^n(x)$, $F^n(x)$ denotes respectively the truth membership, indeterminate membership and false membership degree of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set P.

Definition 2.5: Neutrosophich hesitant fuzzy set

Let X be a fixed set, a neutrosophic hesitantfuzzy set [39] (NHFS) on X is defined as:

$M=\{<x,T(x),I(x),F(x)>|x \in X\}$, where $T(x) = \{\alpha|\alpha \in T(x)\}$, $I(x) = \{\beta|\beta \in I(x)\}$ and $F(x) = \{\gamma|\gamma \in F(x)\}$ are the three sets of some different values in the interval [0, 1], which represent the possible truth-membership hesitant degree, indeterminacy-membership hesitant degree, and falsity-membership hesitant degree of the element $x\epsilon X$ to the set M, and satisfies the following conditions:

$\alpha\epsilon[0,1], \beta\epsilon[0,1], \gamma\epsilon[0,1]$ and $0 \leq sup\,\alpha^+ + sup\beta^+ + sup\gamma^+ \leq 3$ where $\alpha^+ = \bigcup_{\alpha \in T(x)} max\{\alpha\}, \beta^+ = \bigcup_{\beta \in I(x)} max\beta$ and $\gamma^+ = \bigcup_{\gamma \in F(x)} max\{\gamma\}$ for $x \in X$.

The $triplet\ m = \{T(x), I(x), F(x)\}$ is called a neutrosophic hesitant fuzzy element (NHFE) which is the basic unit of the NHFS and is denoted by the symbol $m=\{T, I, F\}$.

Definition 2.6: Interval neutrosophic hesitant fuzzy set

Let X be a nonempty fixed set, an Interval neutrosophic hesitant fuzzy set [67] on X is defined as :

$P = \{\langle x, T(x), I(x), F(x)\rangle | x \in X\}$.

Here $T(x), I(x)$ and $F(x)$ are sets of some different interval values in [0, 1], which denotes respectively the possible truth-membership hesitant degree, indeterminacy-membership hesitant degree, and falsity-membership hesitant degree of the element $x \in \Omega$ to the set P. Then, $T(x)=\{\tilde{\alpha}\,|\tilde{\alpha} \in T(x)\}$, where $\tilde{\alpha} = [\tilde{\alpha}^L, \tilde{\alpha}^U]$ is an interval number; $\tilde{\alpha}^L = inf\,\tilde{\alpha}$ and $\tilde{\alpha}^U = sup\tilde{\alpha}$ represents the lower and upper limits of $\tilde{\alpha}$, respectively; $I(x) = \{\tilde{\beta}|\tilde{\beta} \in I(x)\}$, where $\tilde{\beta} = [\tilde{\beta}^L, \tilde{\beta}^U]$ is an interval number; $\tilde{\beta}^L = \inf\tilde{\beta}$ and $\tilde{\beta}^U = \sup\tilde{\beta}$ represents the lower and upper limits of $\tilde{\beta}$, respectively; $F(x)= \{\tilde{\gamma}|\tilde{\gamma} \in F(x)$, where $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an intervalnumber; $\tilde{\gamma}^L = inf\tilde{\gamma}$ and, $\tilde{\gamma}^U = sup\tilde{\gamma}$ represents the lower and upper limits of $\tilde{\gamma}$, respectively and satisfied the condition

$0 \leq sup\tilde{\alpha}^+ + sup\tilde{\beta}^+ + sup\tilde{\gamma}^+ \leq 3$

where $\tilde{\alpha}^+ = \bigcup_{\tilde{\alpha} \in T(x)} max\{\tilde{\alpha}\}, \tilde{\beta}^+ = \bigcup_{\tilde{\beta} \in I(x)} max\{\tilde{\beta}\}\ and \tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in F(x)} max\{\tilde{\gamma}\}\ for x \in X$.

The triplet $\tilde{p} = \{T(x), I(x), F(x)\}$ is called an interval neutrosophic hesitant fuzzy element or simply INHFE, which is denoted by the symbol $\widetilde{p} = \{T, I, F\}$.

Definition 2.7 Triangular fuzzy neutrosophic sets

Let X be the finite universe and F [0, 1] be the set of all triangular fuzzy numbers on [0, 1]. A triangular fuzzy neutrosophicset (TFNS) [68] P *with* $T_P(x):X\rightarrow F[0,1], I_P:X\rightarrow [0,1]$ and $F_P:X\rightarrow$ in X is defined as:

$P=\{<x:T_P(x),I_P(x),F_p(x)>,x\epsilon X\}$,

where $T_P(x):X\rightarrow F[0,1]$, $I_P:X\rightarrow [0,1]$ and $F_P:X\rightarrow [0,1]$. The triangular fuzzy numbers $T_P(x) =(T_P^1, T_P^2, T_P^3)$, $I_P(x)=(I_P^1, I_P^2, I_P^3)$ and $F_P(x) =(F_P^1, F_P^2, F_P^3)$, respectively, denotesrespectively the possible truth-membership, indeterminacy-membership and a falsi-ty-membership degree of x in P and for every $x\in X$

$0\leq T_P^3(x) + I_P^3(x) + F_P^3(x) \leq 3.$

The triangular fuzzy neutrosophic value (TFNV)P is symbolized by

$<(l,m,n),(p,q,r),(u,v,w)>$where,$(T_P^1(x), T_P^2(x), T_P^3(x)) = (l, m, n)$,$\left(I_P^1(x), I_P^2(x), I_P^3(x)\right) = (p, q, r)$ and $(F_p^1(x), F_p^2(x), F_p^3(x))$ = (u,v,w).

Definition2.8Neutrosophic soft set

Let V be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P(V)$ denote the set of all neutrosophic sets of V. The collection (F, A) is termed to be the soft neutrosophic set [37] over V, where F is a mapping given by $F : A \rightarrow P(V)$.

Definition 2.9 Neutrosophic cubic set

Let U be the space of points with generic element in U denoted by $u\in U$. A neutrosophic cubic set [51]in U defined as $\ddot{\mathrm{N}} = \{< u, A(u), \lambda(u) >: u\in U\}$ in which $A(u)$ is the interval valued neutrosophic set and $\lambda(u)$ is the neutrosophic set in U. A neutrosophic cubic set in U denoted by $\ddot{\mathrm{N}} = <A, \lambda>$. We use $C\ddot{N}(U)$ as a notation which implies that collection of all neutrosophic cubic sets in U.

Definition 2.10 Rough Neutrosophic Sets

Let X be a non empty set and R be an equivalence relation on X. Let P be a neutrosophic set in Y with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (X, R) denoted by $\underline{L}(P)$ *and* $\overline{L}(P)$ are respectively defined as follows:

$$\underline{L}(P)=\left\langle <x,T_{\underline{L}(P)}(x),I_{\underline{L}(P)}(x),F_{\underline{L}(P)}(x)> / y\in [x]_R, x\in X\right\rangle,$$

$$\overline{L}(P)=\left\langle <x,T_{\overline{L}(P)}(x),I_{\overline{L}(P)}(x),F_{\overline{L}(P)}(x)> / y\in [x]_R, x\in X\right\rangle,$$

$T_{\underline{L}(P)}(x)=\wedge_{y\in[x]_R} T_P(y)$,

$I_{\underline{L}(P)}(x)=\vee_{y\in[x]_R} I_P(y)$,

$F_{\underline{L}(P)}(x)=\vee_{y\in[x]_R} F_P(y)$,

$T_{\overline{L}(P)}(x)=\vee_{y\in[x]_R} T_P(y)$,

$I_{\overline{L}(P)}(x)=\wedge_{y\in[x]_R} I_P(y)$,

$F_{\overline{L}(P)}(x)=\wedge_{y\in[x]_R} F_P(y)$

So, $0\leq \sup T_{\underline{L}(P)}(x) + \sup I_{\underline{L}(P)}(x) + \sup F_{\underline{L}(P)}(x) \leq 3.$

$0\leq \sup T_{\overline{L}(P)}(x) + \sup I_{\overline{L}(P)}(x) + \sup F_{\overline{L}(P)}(x) \leq 3.$

Here $\vee$ and $\wedge$ denotes "max" and "min'' operators respectively. $T_P(y)$, $I_P(y)$ and $F_P(y)$ are the membership, indeterminacy and non-membership function of y with respect to P and also $\underline{L}(P)$ and $\overline{L}(P)$ are two neutrosophic sets in X.

Therefore, NS mapping $\underline{L}$, $\overline{L}:L(X) \rightarrow L(X)$ are, respectively, referred to as the lower and the upper rough NS approximation operators, and the pair $(\underline{L}(P), \overline{L}(P))$ is called the rough neutrosophic set [42] in (Y, R).

Definition 2.11Refined Neutrosophic Sets

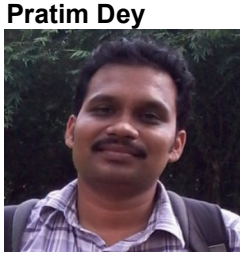
Let$X$ be a universe. A neutrosophic refined set (NRS) [49]A on X can be defined as follows:

$$A = \left\{ \begin{matrix} < x,(T_A^1(x),T_A^2(x),...,T_A^p(x)),(I_A^1(x),I_A^2(x),...,I_A^p(x)), \\ (F_A^1(x),F_A^2(x),...,F_A^p(x)) > \end{matrix} \right\}$$

Here, $T_A^1(x), T_A^2(x),..., T_A^p(x) : X \rightarrow [0,1]$, $I_A^1(x), I_A^2(x),..., I_A^p(x) : X \rightarrow [0,1]$, and $F_A^1(x), F_A^2(x),..., F_A^p(x) : X \rightarrow [0,1]$. For any $x \in X$ $\left(T_A^1(x), T_A^2(x),..., T_A^p(x)\right)$, $\left(I_A^1(x), I_A^2(x),..., I_A^p(x)\right)$ and $\left(F_A^1(x), F_A^2(x),..., F_A^p(x)\right)$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Section 3 The contribution of the selected Indian researchers

3.1 Dr. Partha Pratim Dey



Dr. Partha Pratim Dey was born at Chak, P. O.-Islampur, Murshidabad, West Bengal, India, PIN-742304. Dr. Dey qualified CSIR-NET-Junior Research Fellowship (JRF) in 2008. His paper entitled"Fuzzy goal programming for multilevel linear fractional programming problem"coauthored with Surapati Pramanik was awarded as the best paper in West Bengal State Science and Technology Congress (2011) in mathematics. He obtained Ph. D. in Science from Jadavpur University, India in 2015.Title of his Ph. D. Thesis [70] is:"Some studies on linear and non-linear bi-level programming problems in fuzzy envieonment``. He continues his research in the feild of fuzzy multi-criteria decision making and extends them in neutrosophic environment. Curently, he is an assistant teacher of Mathematics in Patipukur Pallisree Vidyapith, Patipukur, Kolkata-48. His research interest includes decision making in neutrosophic environemnt and optimization.

Contribution:

In 2015, Dey, Pramanik, and Giri [71] proposed a novel MADM strategy based on extended grey relation analysis (GRA) in interval neutrosophic environment with unknown weight of the attributes. Maximizing deviation method is employed to determine the unknown weight information of the atributes. Dey et al. [71] also developed linguistic scale to transform linguistic variable into interval neutrosophic values. They employed the developed strategy for dealing with practical problem of selecting weaver for Khadi Institution. Partha Pratim Dey, coming from a weaver family, is very familiar with the parameters of weaving and criteria of selection of weavers. Several parameters are defined by Dey et al. [71] to conduct the study.

Dey et al. [72] proposed a TOPSIS strategy at first in single valued neutrosophic soft expert set environmnet in 2015. Dey et al. [72] determined the weights of the parameters by employing maximizing deviation method and demonstrated an illustrative example of teacher selection problem. According to Google Scholar Citation, this paper [72] has been cited by 15 studies so far.

In 2015, Dey et al. [73] established TOPSIS startegy in generalized neutrosophic soft set environmnet and solved an illustrative MAGDM problem. In neutrosophic soft set environment**,** Dey et al. [74] grounded a new MADM strategy based on grey relational projection technique.

In 2016, Dey et al. [75] developed two new strategies for solving MADM problems with interval-valued neutrosophic assessments. The empolyed measures [75] are namely, i) weighted projection measure and ii) angle cosine and projection measure. Dey et al. [76] defined Hamming distance function and Euclidean distance function between bipolar neutrosophic sets. In the same study, Dey et al. [76] defined bipolar neutrosophic relative positive ideal solution (BNRPIS) and neutrosophic relative negative ideal solution(BNRNIS) and developed an MADM strategy in bipolar neutrosophic environemnt.

Deyet et al. [77] presented a GRA strategy for solving MAGDM problem under neutrosophic soft environment and solved an illustrative numerical example to show the effectiveness of the proposed strategy.

In 2016, Dey et al. [78] discussed a solution strategy for MADM problems with interval neutrosophic uncertain linguistic information through extended GRA method. Dey et al. [78] also proposed Euclidean distance between two interval neutrosophic uncertain linguistic values.

Pramanik, Dey, Giri, and Smarandache [79] defined projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets in 2017 and proved their basic properties. In the same study [79], the same authors developed three new MADM strategies based on the proposed projection measures. They validated their result by solving a numerical example of MADM.

In 2017, Pramanik, Dey, Giri, and Smarandache [80] defined some operation rules for neutrosophic cubic sets and introduced the Euclidean distance between them.nThe authors also defined neutrosophic cubic positive and negative ideal solutions and established a new MADM strategy. In 2018, Dey, Pramanik, Ye and Smarandache [81] introduced cross entropy and weighted cross entropy measures for bipolar neutro-

sophic sets and interval bipolar neutrosophic sets and proved their basic properties. The authors also developed two new multi-attribute decision-making strategies in bipolar and interval bipolar neutrosophic set environment. The authors solved two illustrative numerical examples and compared the obtained results with existing strategies to demonstrate the feasibility, applicability, and efficiency of their strategies.

Pramanik, Dey and Giri [82] defined hybrid vector similarity measure between single valued refined neutrosophic sets (SVRNSs) and proved their basic properties and developed an MADM strategy and employed them to solve an illustrative example of MADM in SVRNS environment.

Pramanik, Dey and Smarandache [83] defined the correlation coefficient measure $Cor\ (L_1, L_2)$ between two interval bipolar neutrosophic sets (IBNSs) L_1, L_2 and proved the following properties:

(1) $Cor\ (L_1, L_2) = Cor\ (L_2, L_1)$;

(2) $0 \leq Cor\ (L_1, L_2) \leq 1$;

(3) $Cor\ (L_1, L_2) = 1$, if $L_1 = L_2$.

In the same study, the authors defined weighted correlation coefficient measure $Cor_w(L_1, L_2)$ between two IBNSs L_1, L_2 and established the following properties:

(1) $Cor_w(L_1, L_2) = Cor_w\ (L_2, L_1)$;

(2) $0 \leq Cor_w(L_1, L_2) \leq 1$;

(3) $Cor_w(L_1, L_2) = 1$, if $L_1 = L_2$.

The authors [83] also developed a novel MADM straegy based on weighted correlation coefficient measure and empolyed to solve an investment problem and compared the solution with existing startegies.

Pramanik, Dey, and Smarandache [84] defined Hamming and Euclidean distances measures, similarity measures based on maximum and minimum operators between two IBNSs and proved their basic properties. In the same research, Pramanik et al. [84] deveolped a novel MADM strategy in IBNS environment.

In fuzzy environment, work of Dey and Pramanik [85] obtained the best paper award in Mathematics in 2011 at 18th West Bengal State Science & Technology Congress Tilte of the paper was:‘ Fuzzy goal programming for multilevel linear fractional programming problems’.

In 2015, Dr. Dey obtained “*Diploma Certificate”* from *Neutrosophic Science InternationalAssociation (NISA)* for his outstanding performance in neutrosophic research. He was awarded the certificate of outstanding contribution in reviewing for the International Journal “Neutrosophic Sets and Systems“. His works in neutrosophics draw much attention of the researchers international level. According to “*ResearchGate*’’ a social networking site for scientists and researchers, citation of his research exceeds 200. He is an active member of ‘‘*Indian society for neutrosophic study*’’.

Dr. Dey is very much intersted in neutrosophic study. He continues his research work with great mathematician like Prof. Florentin Smarandache and Prof. Jun Ye.

3.2 Kalyan Mondal

Kalyan Mondal was born at Shantipur, Nadia, West Bengal, India, Pin-741404. He qualified CSIR-NET-Junior Research Fellowship (JRF) in 2012. He is a research scholar in Mathematics of Jadavpur University, India since 2016. Title of his Ph. D. thesis is: “Some decision making models based on neutrosophic strategy”. His paper entiled “*MAGDM based on contra-harmonic aggregation operator in neutrosophic number (NN) environme*nt’’ coauthored with Surapsati Pramanik and Bibhas C. Giri was awarded outstanding paper in West Bengal State Science and Technology Congress (2018) in mathematics. He continues his research in the field neutrosophic multi-attribute decision making; aggregation operators; soft computing; pattern recognitions; neutrosophic hybrid systems, rough neutrosophic sets, neutrosophic numbers, neutrosophic game theory, neutrosophic algebraic structures. Presently, he is an assistant teacher of Mathematics in Birnagar High School (HS) Birnagar, Ranaghat, Nadia, Pin-741127, West Bengal, India.

Contribution:

In 2014, Mondal and Pramanik [86]initiated to study teacher selection problem using neutrosophic logic.

Pramanik and Mondal [87] defined cosine similarity measure for rough neutrosophic sets as $C_{RNS}(A, B)$ between two rough neutrosophic sets A, B and established the following properties:

(1) $C_{RNS}(A, B) = C_{RNS}\ (B, A)$;

(2) $0 \leq C_{RNS}(A, B) \leq 1$;

(3) $C_{\mathrm{RNS}}(A, B) = 1$, iff $A = B$.

In the same study, Pramanik and Mondal [87] applied cosine similarity measure for medical diagnosis.

Mondal et al. [88] proposed a rough cotangent similarity measure in 2015 and studied some of its basic properties. The authors demonstrated an application of cotangent similarity measure of rough neutrosophic sets for medical diagnosis.

Pramanik and Mondal [89] introduced interval neutrosophic MADM strategy with completely unknown attribute weight information based on extended grey relational analysis.

In 2015, Mondal and Pramanik [90] presents rough neutrosphic MADM strategy based on GRA. They also extended the neutrosophic GRA strategy to rough neutrosophic GRA strategy and applied it to MADM problem. The authors first defined accumulated geometric operator to transform rough neutrosophic number (neutrosophic pair) to single valued neutrosophic number.

In 2015, Mondal and Pramanik [91] presented a neutrosophic MADM strategy for school choice problem. The authors used five criteria to modeling the school choice problem in neutrosophic environment.

In 2015, Mondal and Prammanik [92] defined cotangent similarity measure for neutrosophic sets as $COT_{NRS}(N, P)$ between two refined neutrosophic sets N, P and established the following properties:

(1) $COT_{NRS}(N, P) = COT_{NRS}(P, N)$;

(2) $0 \leq COT_{NRS}(N, P) \leq 1$;

(3) $COT_{NRS}(P, N) = 1$, if $P = N$.

In the same study, Mondal and Pramanik [92] presented an application of cotangent similarity measure of neutrosophic single valued sets in a decision making problem for educational stream selection.

Mondal and Pramanik [93] also defined rough accuracy score function and proved their basic properties. The authors also introduced entropy based weighted rough accuracy score value. The authors developed a novel rough neutrosophic MADM startegy with incompletely known or completely unknown attribute weight information based on rough accuracy score function.

Pramanik and Mondal [94] presented rough Dice and Jaccard similarity measures between rough neutrosophic sets. The authors proposed weighted rough Dice and Jaccard similarity measures, and proved their basic properties. The authors presented an application of rough neutrosophic Dice and Jaccard similarity measures in medical diagnosis.

Mondal and Pramanik [95] defined tangent similarity measure and proved their basic properties. In the same study, Mondal and Pramanik developed a novel MADM strategy for MADM problems in SVNS environment. The authors resented two illustrattive exaxmples, namely selection of educational stream and medical diagnosis to demonstrate the feasibility, and applicability of the proposed MADM strategy.

Mondal and Pramanik [96] studied the quality clay-brick selection strategy based on MADM with single valued neutrosophic GRA.The authors used neutrosophic grey relational coefficient on Hamming distance between each alternative to ideal neutrosophic estimates reliability solution and ideal neutrosophic estimates unreliability solution. They also used neutrosophic relational degree to determine the ranking order of all alternatives.

In 2015, Mondal and Pramanik [97] defined a refined tangent similarity measure strategy of refined neutrosophic sets and proved its basic properties. They presented an application of refined tangent similarity measure in medical diagnosis.

Mondal and Pramanik [98] introduced cosine, Dice and Jaccard similarity measures of interval rough neutrosophic sets and proved their basic properties. They developed three MADM strategies based on interval rough cosine, Dice and Jaccard similarity measures and presented an illustrative example, namely selection of best laptop for random use.

In 2016, Mondal and Pramanaik [47] defined rough tri-complex similarity measure in rough neutrosophic environment and proved its basic properties. In the same study, Mondal and Pramnaik [47] developed a novel MADM strategy for dealing with MADM problem in rough tri-complex neutrosophic envioronment. Mondal, Pramanik, and Smarandache [48] introduced the rough neutrosophic hyper-complex set and the rough neutrosophic hyper-complex cosine function in 2016, and proved their basic properties. They also defined the rough neutrosophic hyper-complex similarity measure and proved their basic properties. They also developed a new MADM strategy to deal with MADM problems in rough neutrosophic hyper-complex set environment. They presented a hypothetical application to the selection problem of best candidate for marriage for Indian context.

Mondal, Pramanik, and Smarandache [99] defined rough trigonometric Hamming similarity measures and proved their basic properties. In the same study, Mondal et al. [99] developed a novel MADM strategies to solve MADM problems in rough

neutrosophic environment. The authors provided an application, namely selection of the most suitable smart phone for rough use.

In 2017, Mondal, Pramanik and Smarandache [100] developed a new MAGDM strategy by extending the TOPSIS strategy in rough neutrosphic environment, called rough neutrosophic TOPSIS strategy for MAGDM. They also proposed rough neutrosophic aggregate operator and rough neutrosophic weighted aggregate operator. Finally, the authors solved a numerical example to demonstrate the applicability and effectiveness of the proposed TOPSIS startegy.

Mondal, Pramanik, Giri and Smarandache [101] proposed neutrosophic number harmonic mean operator (NNHMO) and neutrosophic number weighted harmonic mean operator NNWHMO and cosine function to determine unknown criterion weights in neutrosophic number (NN) environment. The authors developed two strategies of ranking NNs based on score function and accuracy function. The authors also developed two novel MCGDM strategies based on the proposed aggregation operators. The authors solved a hypothetical case study and compared the obtained results with other existing strategies to demonstrate the effectiveness of the proposed MCGDM strategies. The significance of these stratigies is that they combine NNs with harmonic aggregation operators to cope with MCGDM problem.

In 2018, Mondal, Pramanik and Giri [102] inroduced hyperbolic sine similarity measure and weighted hyperbolic sine similarity measure namely, SVNHSSM(A, B) for SVNSs. They proved the following basic properties.

1. $0 \leq$ SVNHSSM(A, B) ≤ 1
2. SVNHSSM(A, B) = 1 if and only if $A = B$
3. SVNHSSM (A, B) = SVNHSSM(B, A)
4. If R is a SVNS in X and $A \subset B \subset R$ then SVNHSSM(A, R) $\leq$ SVNHSSM(A, B) and SVNHSSM(A, R) $\leq$ SVNHSSM(B, R).

The authors also defined weighted hyperbolic sine similarity measure for SVNS namely, SVNWHSSM(A, B) and proved the following basicproperties.

1. $0 \leq$ SVNWHSSM(A, B) ≤ 1
2. SVNWHSSM (A, B) = 1 if and only if $A = B$
3. SVNWHSSM (A, B) = SVNWHSSM(B, A)
4. If R is a SVNS in X and $A \subset B \subset R$ then SVNWHSSM (A, R) $\leq$ SVNWHSSM(A, B) and SVNWHSSM (A, R) $\leq$ SVNWHSSM (B, R).

The authors defined compromise function to determine unknown weight of the attributes in SVNS environment. The authors developed a novel MADM strategy based on the proposed weighted similarity measure. Lastly, the authors solved a numerical example and compared the obtained results with the existing strategies to demonstrate the effectiveness of the proposed MADM strategy.

Mondal, Pramanik, and Giri [103] defined tangent similarity measure and proved its properties in interval valued neutrosophic environment. The authors developed a novel MADM strategy based on the proposed tangent similarity measure in interval valued neutrosophic environment. The authors also solved a numerical example namely, selection of the best investment sector for an Indian government employee. The authors also presented a comparative analysis.

Mondal et al. [104] employed refined neutrosophic set to express linguistic variables. The authors proposed linguistic refined neutrosophic set. The authors developed an MADM strategy based on linguistic refined neutrosophic set. The authors also proposed an entropy method to determine unknown weight of the criterion in linguistic neutrosophic refined set environment. They presented an illustrative example of constructional spot selection to show the feasubility and applicability of the proposed strategy.

Mr. Kalyan Mondal is a young and hardworking researcher in neutrosophic field. He acts as an area editor of international journal,"*Journal of New Theory"* and acts as a reviewer for different international peer reviewed journals. In 2015, Mr. Mondal was awarded Diploma certificate from *Neutrosophic Science InternationalAssociation (NISA)* for his outstanding performance in neutrosophic research. He was awarded the certificate of outstanding contribution in reviewing for the International Journal "*Neutrosophic Sets and Systems*''. His works in neutrosophics draw much attention of the researchers at international level. According to "*Researchgate*'', citation of his research exceeds 430.

3.3 Dr. Pranab Biswas

Pranab Biswas obtained his Bachelor of Science degree in Mathematics and Master degree in Applied Mathematics from University of Kalyani. He obtained Ph. D. in Science from Jadavpur University, India. Title of his thesis is "Multi-attribute decision making in neutrosophic environment".

He is currently an assistant teacher of Mathematics. His research interest includes multiple criteria decision making, aggregation operators, soft computing, optimization, fuzzy set, intuitionistic fuzzy set, neutrosophic set.

Contribution:

In 2014, Biswas, Pramanik and Giri [105] proposed entropy based grey relational analysis strategy for MADM problem with single valued neutrosophic attribute values. In neutrosophic environment, this is the first case where GRAwas applied to solve MADM problem. The authors also defined neutrosophic relational degree. Lastly, the authors provided a numerical example to show the feasibility and applicability of the developed strategy.

In 2014, Biswas et al. [106] introduced single –valued neutrosophic MADM strategy with incompletely known and completely unknown attribute weight information based on modified GRA.The authors also solved an optimization model to find out the completely unknown attribute weight by ustilizing Lagrange function. At the end, the authors provided an illustrative example to show the feasibility, practicalitry and effectiveness of the proposed strategy.

Biswas et al. [69] introduced a new strategy called "Cosine similarity based MADM with trapezoidal fuzzy neutrosophic numbers".The authors also established expected interval and the expected value for trapezoidal fuzzy neutrosophic number and cosine similarity measure of trapozidal fuzzy neutrosophic numbers.

In 2015, Biswas et al. [107] extended TOPSIS strategy for MAGDM in neutrosophic environment. In the study, rating values of alternative are expressed by linguistic terms such as *Good*, *Very Good*, *Bad*, *Very Bad*, etc. and these terms are scaled with single-valued neutrosophic numbers. Single-valued neutrosophic set-based weighted averaging operator is used to aggregate all the individual decision maker's opinion into one common opinion for rating the importance of criteria and alternatives. The authors provided an illustrative example to demonstrate the proposed TOPSIS strategy.

Biswas et al. [108] further extened the TOPSIS strategy for MAGDM in single-valued neutrosophic environment. The authors developed a non-linear programming based strategy to study MAGDM problem. In the same study, the authors converted the single valued neutrosophic numbers into interval numbers. The authors employed nonlinear programming model to determine the relative closeness co-efficient intervals of alternatives for each decision maker. Then, the closeness co-efficient intervals of each alternative are aggregated according to the weight of decision makers. Further, the authors developed a priority matrix with the aggregated intervals of the alternatives. The authors obtained the ranking order of all alternatives by computing the optimal membership degrees of alternatives with the ranking method of interval numbers. Finally, the authors presented an illustrative example to show the effectiveness of the proposed strategy.

In 2015, Pramanik, Biswas, and Giri [109] proposed two new hybrid vector similarity measures of single valued and interval neutrosophic sets by hybriding the concept of Dice and cosine similarity measures.The authors also proved their basic properties. The authors also presented their applications in multi-attribute decision making in neutrosophic environment.

Biswas et al. [110] proposed triangular fuzzy number neutrosophic sets by combining triangular fuzzy number with single valued neutrosophic set in 2016. Biswas et al. [110] also defined some of its operational rules. The authors defined triangular fuzzy number neutrosophic weighted arithmetic averaging operator and triangular fuzzy number neutrosophic weighted geometric averaging operator to aggregate triangular fuzzy number nuetrosophic set. The authors also established some of their properties of the proposed operators. The authors also presented an MADM strategy to solve MADM in triangular fuzzy number neutrosophic set environment.

In 2016, Biswas et al. [111] defined score value, accuracy value, certainty value, and normalized Hamming distance of single valued neutrosophic hesitant fuzzy sets.The authors also defined positive ideal solution and negative ideal solution by score value and accuracy value. The authors calculated the degree of grey relational coefficent between each alternative and ideal alternative. The authors also determined a relative closeness coefficient to obtain the ranking order of all alternatives. Finally, the authors provided an illustrative example to show the

validity and effectiveness of the proposed grey relational analysis based MADM strategy in single valued neutrosophic hesitant fuzzy set environment.

Biswas, Pramanik, and Giri [112] proposed a class of distance measures for single-valued neutrosophic hesitant fuzzy sets in 2016 and proved their properties with variational parameters. The authors applied weighted distance measures to calculate the distances between each alternative and ideal alternative in the MADM problems. The authors developed a MADM strategy based on the proposed distance functions in single valued neutrosophic hesitant fuzzy set environment. The authors provided an illustrative example to verify the proposed strategy and to show its fruitfulness. The authors also compared the proposed strategy with other existing startegies for solving MADM in single valued neutrosophic hesitant fuzzy set environment.

Biswas et al. [113] introduced single-valued trapezoidal neutrosophic number (SVTrNN), which is a special case of single-valued neutrosophic number and developed a ranking method for ranking SVTrNNs. The authors presented some operational rules as well as cut sets of SVTrNNs. The authors defined the value and ambiguity indices of truth, indeterminacy, and falsity membership functions of SVTrNNs. Using the proposed ranking strategy and proposed indices, the authors developoed a new MADM strategy to solve MADM problem in which the ratings of the alternatives over the attributes are expressed in terms of TrNFNs. Finally, the authors provided an illustrative example to demonstrate the validity and applicability of the proposed MADM strategy with SVTrNNs.

In 2016, Biswas et al.[114] introduced the concept of SVTrNN in the form:

$\tilde{A}_1 = \langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$,where $a_{11}, a_{21}, a_{31}, a_{41}$, $b_{11}, b_{21}, b_{31}, b_{41}, c_{11}, c_{21}, c_{31}, c_{41}$ are real numbers and satisfy the inequality $c_{11} \leq b_{11} \leq a_{11} \leq c_{21} \leq b_{21} \leq a_{21} \leq a_{31} \leq b_{31} \leq c_{31} \leq a_{41} \leq b_{41} \leq c_{41}$.

The authors defined some arithmetical operational rules. The authors also defined value index and ambiguity index of SVTrNNs and established some of their properties. The authors developed a ranking strategy with the proposed indicess to rank SVTrNNs. The authors developed a new MADM strategy to solve MADM problems in SVTrNN environment.

Biswas et al. [115] extended the TOPSIS strategy of MADM problems in single-valued trapezoidal neutrosophic number environment. In their study, the attribute values are expressed in terms of single-valued trapezoidal neutrosophic numbers. The authors deal with the situation where the weight information of attribute is incompletely known or completely unknown. The authors developed an optimization model using maximum deviation strategy to obtain the weight of the attributes. The authors also illustrated and validated the proposed TOPSIS strategy by solving a numerical example of MADM problems.

Biswas et al. [116] introduced a new neutrosophic numbers called interval neutrosophic trapezoidal number (INTrN) characterized by interval valued truth, indeterminacy, and falsity membership degrees and defined some arithmetic operations on INTrNs, and normalized Hamming distance between INTrNs. In the same study, Biswas et al. [116] developed a new MADM strategy, where the rating values of alternatives over the attributes and the importance of weight of attributes assume the form of INTrNs. Biswas et al. [116] employed the entropy strategy to determine thr attribute weight and then used it to calculate aggregated weighted distance measure and determined ranking order of alternatives with the help of aggregated weighted distance measures. Biswas et al. [116] also solved an illustrative example to show the feasibility, applicability and effectiveness of the proposed strategy.

Dr. Biswas's work [117] obtained outstanding paper award at "Second Regional Science and Technology Congress, 2017'' held at University of Kalyani, Nadia, West Bengal, India. His resesrch interest includes fuzzy, intuitionistic fuzzy and neutrosophic decision making.

Dr. Pranab Biswas is a young and hardworking researcher in neutrosophic field. In 2015, Dr. Biswas was awarded "Diploma Certificate" from *Neutrosophic Science International Association (NISA)* for his outstanding performance in neutrosophic research. He was awarded the certificate of outstanding contribution in reviewing for the International Journal "Neutrosophic Sets and Systems'' in 2018. According to "*Researchgate*'', citation of his research exceeds 375. Research papers of Biswas et al. [105, 112] received the best paper award from "Neutrosophic Sets and Systems'' for volume 2, 2014 and volume 12, 2016. His works in neutrosophics draw much attention of the researchers in national as well international level. His Ph. D. thesis entilted:"Multi-attribute decision making in neutrosophic environment" was awarded "*Doctorate of Neutrosophic theory*" by Indian Society for Neutrosophic Study (ISNS) with sponsorship by Neutrosophic Science International Association (NSIA).

3.4 Dr.Durga Banerjee

Durga Banerjee passed M. Sc. from Jadavpur University in 2005. In 2017, D. Banerjee obtained Ph. D. Degree in Science from Jadavpur University. Her research interest includes operations research, fuzzy optimization, and neutrosophic decision making. Title of her Ph. D. Thesis [118] is: "Some studies on decision making in an uncertain environment''. Her Ph. D. thesis comprises of few chapters dealing with MADM in neutrosophic environment.

Contribution:

In 2016, Pramanik, Banerjee, and Giri [119] introduced refined tangent similarity measure.The authors presented an MAGDM model based on tangent similarity measure of neutrosophic refined set. The authors also introduced simplified form of tangent similarity measure. The authors defined new ranking method based on refined tangent similarity measure. Lastly, the authors solved a numerical example of teacher selectionin in neutrosophic refined set environment to see the effectiveness of the proposed strategy.

In 2016, Banerjee et al.[120] developed TOPSIS startegy for MADM in refined neutrosophic environment. The authors also provided a numerical example to show the feasibility and applicability of the proposed TOPSIS strategy.

In 2017, Banerjee, Pramanik, Giri and Smarandache [121] at first developed an MADM strategy in neutrosophic cubic set environment using grey relational analysis. The authors discussed about positive and negative grey relational coefficients,and weighted grey relational coefficients, Hamming distances for weighted grey relational coefficients and standard grey relational coefficient.

Her Ph. D. thesis [118] entilted:"Multi-attribute decision making in neutrosophic environment" was awarded "*Doctorate of Neutrosophic theory*" by the Indian Society for Neutrosophic Study (ISNS) with sponsorship by Neutrosophic Science International Association (NSIA). According to "*Researchgate*'', citation of his research exceeds 55.

3.5 Shyamal Dalapati

Shyamal Dalapati qualified CSIR-NET-Junior Research Fellowship (JRF) in 2017. He is a research scholar in Mathematics at the Indian Institute of Engineering Science and Technology (IIEST), Shibpur, West Bengal, India.Title of his Ph. D. thesis is:"Some studies on neutrosophic decision making". He continues his research in the field of neutrosophic multi attribute group decision making; neutrosophic hybrid systems; neutrosophic soft MADM . Curently, he is an assistant teacher of Mathematics His research interest includes decision making in neutrosophic environemnt and optimization.

Contribution:

In 2016, Dalapati and Pramanik [122] defined neutrosophic soft weighted average operator.They determined the order of the alternatives and identify the most suitable alternative based on grey relational coefficient. They also presented a numerical example of logistics center location selection problem to show the effectiveness and applicability of the proposed strategy.

Dalapati,Pramanik, and Roy [123] proposed modeling of logistics center location problem using the score and accuracy function, hybrid-score-accuracy function of SVNNs and linguistic variables under single-valued neutrosophic environment, where weight of the decision makers are completely unknown and the weight of criteria are incompletely known.

Dalapati, Pramanik, Alam, Roy, and Smaradache [124] defined IN-cross entropy measure in INS environment in 2017. The authors proved the basic properties of the cross entropy measure. The authors also defined weighted IN- cross entropy measure and proved its basic properties. They also introduced a novel MAGDM strategy based on weighted IN-cross entropy. Finally, the authors solved a MAGDM problem to show the feasibility and efficiency of the proposed MAGDM strategy.

Pramanik, Dalapati, Alam, and Roy [125] defined TODIM strategy in bipolar neutrosophic set environment to handle MAGDM. The authors proposed a new strategy for solving MAGDM problems. The authors also solved an MADM problem to show the applicability and effectiveness of the proposed startegy.

Pramanik, Dalapati, Alam, and Roy [126] introduced the score and accuracy functions for neutrosophic cubic sets and prove their basic properties in 2017. The authors developed a new strategy for ranking of neutrosophic cubic numbers based on the score and accuracy functions. The authors first developed a TODIM (Tomada de decisao interativa e multicritévio) stratey in the neutrosophic cubic set (NCS) environment strategy. The authors also solved an MAGDM problem to show the applicability and effectiveness of the developed strategy. Lastly, the authors conducted a comparative study to show the usefulness of proposed strategies.

In 2018, Pramanik, Dalapati, Alam, and Roy [127]extended the traditional VIKOR strategy to NC-VIKOR strategy and developed an NC-VIKOR based MAGDM strategy in neutrosophic cubic set environment. The authors defined the basic concept of neutrosophic cubic set. Then, the authors introduced neutrosophic cubic number weighted averaging operator and applied it to aggregate the individual opinion to one group opinion. The authors presented an NC-VIKOR based MAGDM strategy with neutrosophic cubic set. They also presented a sensitivity analysis. Finally, the authors solved an MAGDM problem to show the feasibility and efficiency of the proposed MAGDM strategy.

Pramanik, Dalapati, Alam, and Roy [128] extended the VIKOR strategy to MAGDM with bipolar neutrosophic environment. The authors introduced the bipolar neutrosophic numbers weighted averaging operator and applied it to aggregate the individual opinion to one group opinion. The authors proposed a VIKOR based MAGDM strategy with bipolar neutrosophic set. Lastly, the authors solved an MAGDM strategy to show the feasibility and efficiency of the proposed MAGDM strategy and presented a sensitivity analysis.

Pramanik, Dalapati, Alam, and Roy [129] studied some operations and properties of neutrosophic cubic soft sets.The authors defined some operations such as P-union, P-intersection, R-union, R-intersection for neutrosophic cubic soft sets (NCSSs). The authors proved some theorems on neutrosophic cubic soft sets.The authors also discussed various approaches of internal neutrosophic cubic soft sets (INCSSs) and external neutrosophic cubic soft sets (ENCSSs) and also investigated some of their properties.
Pramanik, Dalapati, Alam, Smarandache, and Roy [130] defined a new cross entropy measure in SVNS environment.The authors also proved the basic properties of the NS cross entropy measure. The authors defined weighted SN-cross entropy measure and proved its basic properties. At first the authors proposed an MAGDM strategy based on NS- cross entropy measure.

Pramanik, Dalapati, Alam, Roy, Smarandache [131] defined similarity measure between neutrosophic cubic sets and proved its basic properties. They developed a new MADM strategy basd on the proposed similarity measure. They also provided an illustrative example for MADM strategy to show its applicability and effectiveness.

Mr. Dalapati's neutrosophic paper [132] was awarded as the outstanding research paper at the "1st Regional Science and Technology Congress, 2016 in mathematics.

Mr. Shamal Dalapati is a young and hardworking researchers in neutrosophic field. In 2017, Mr. Dalapati was awarded "Diploma Certificate" from *Neutrosophic Science InternationalAssociation (NISA)* for his outstanding performance in neutrosophic research. His research articles receive more than sevent citations.

3.6 Prof.Tapan Kumar Roy

Prof. T. K. Roy, Ph. D. in mathematics, is a Professor of mathematics in Indian Institute of Engineering Science and Technology (IIEST), Shibpur. His main research interest includes neutrosophic optimization, neutrosophic game theory, decision making in neutrosophic environment, neutrosophy, etc.

Contribution:

In 2014, Pramanik and Roy [133] presented the framework of the application of game theory to Jammu Kashmir conflict between India and Pakistan. Pramanik and Roy [20] extended the concept of game

theoretic model [133] of the Jammu and Kashmir conflict in neutrosophic environment.
At first, Roy and Das[134] presented multi-objective non –linear programming problem based on neutrosophic optimization technique and its application in Riser design problem in 2015.
Roy, Sarkar, and Dey [133] presented a multi-objective neutrosophic optimization technique and its application to structural design in 2016.

In 2017, Roy and Sarkar [135-138] also presented several applications of neutrosophic optimization technique.

In 2017, Pramanik, Roy, Roy, and Smarandache [139] presented multi criteria decision making using correlation coefficient under rough neutrosophic environment. The authors defined correlation coefficient measure between any two rough neutrosophic sets and also proved some of its basic properties.

In 2018, Pramanik, Roy, Roy, and Smarandache [140] defined projection and bidirectional projection measures between interval rough neutrosophic sets and proved their basic properties. The authors developed two new MADM strategies based on interval rough neutrosophic projection and bidirectional projection measures. Then the authors solved a numerical example to show the feasibility, applicability and effectiveness of the proposed strategies.

In 2018, Pramanik, Roy, Roy, and Smarandache [141] proposed the sine, cosine and cotangent similarity measures of interval rough neutrosophic sets and proved their basic properties. The authors presented three MADM strategies based on proposed similarity measures. To demonstrate the applicability, the authors solved a numerical example. Prof. Roy did research work on decision making in SVNS, INS, neutrosophic hybrid environment [124-132, 139-141] with S. Pramanik, S. Dalapati, S. Alam and Rumi Roy.

His paper [142] together with S. Pramanik and S. Chackrabarti was awarded as the best research paper in 15th West Bengal State Science & Technology Congress, 2008 held on 28th February-29th February, 2008, at Bengal Engineering and Science University, Shibpur.
Prof. Roy is a great motivator and a very hardworking person. He works with Prof. Florentin Smarandache. According to "Googlescholar" his research gets citation over 2635.

3.7Prof.Bibhas C. Giri

Prof. Bibhas C.Giri is a Prof. of mathematics in Jadavpur University. He did his M.S. in Mathematics and Ph. D. in Operations Research both from Jadavpur University, Kolkata, India. His research interests include inventory/supply chain management, production planning and scheduling, reliability and maintenance.
He was a JSPS Research Fellow at Hiroshima University, Japan during the period 2002-2004 and Humboldt Research Fellow at Mannheim University, Germany during the period 2007-2008, Fulbright Senior Research Fellow at Louisiana State University in the year 2012.

Contribution:

Prof. Giri works with S. Pramanik, P. Biswas and P. P. Dey in neutrosophic environment. His neutrosophic paper [143] coauthored with Kalyan Mondal and Surapati Pramanik received the outstanding research paper award at the"1st Regional Science and Technology Congress, 2016 in mathematics. His neutrosophic paper [144] together with Kalyan Mondal and Surapati Pramanik received the best research paper in 25 th West Bengal State Science and Technology Congress 2018 in mathematics. His neutrosophic research work and vast contribution can be found in [71-80, 82, 101-119].

Prof. Giri is a great motivator. According to "*Googlescholar*', his research receives more than 4920 citations having h-index-31 and i-10 index-78.

3.8 Prof. Anjan Mukherjee

Anjan Mukherjee was born in 1955. He completed his B. Sc. and M. Sc. in Mathematics from University of Calcutta and Ph. D. from Tripura University. Currently, he is a Professor and Pro -Vice Chancellor

of Tripura University. Under his guidance, 12 candidates obtained Ph. D. award. He has 30 years of research and teaching experience. His main research interest includes topology, fuzzy set theory, rough sets, soft sets, neutrosophic set, neutrosophic soft set, etc.

Contribution:

In 2014, Anjan Mukherjee and Sadhan Sarkar [145] defined the Hamming and Euclidean distances between two interval valued neutrosophic soft sets (IVNSSs). The authors also introduced similarity measures based on distances between two interval valued neutrosophic soft sets.The authors proved some basic properties of the similarity measures between two interval valued neutrosophic soft sets. They established an MADM strategy for interval valued neutrosophic soft set setting using similarity measures.

Mukherjee and Sarkar [146] also defined several distances between two interval valued neutrosophoic soft sets in 2014. The authors proposed similarity measure between two interval valued neutrosophic soft sets. The authors also proposed similarity measure between two interval valued neutrosophic soft sets based on set theoretic approach. They also presented a comparative study of different similarity measures.

Mukherjee and Sarkar [147]defined several distances between two neutrosophoic soft sets.The authors also defined similarity measure between two neutrosophic soft sets.The authors developed an MADM strategy based on the proposed similarity measure.

Mukherjee and Sarkar [148] proposed a new method of measuring degree of similarity and weighted similarity between two neutrosophic soft sets and studied some properties of similarity measure. Based on the comparison between the proposed strategy [148] and existing strategies introduced by Mukherjee and Sarkar[147], the authors found that the proposed strategy [148] offers strong similarity measure. The authors also proposed a decision making strategy based on similarity measure.

Prof. Anjan Mukherjee evaluated many Ph. D. theses. Among them, the Ph. D. thesis of Durga Banerjee [118] dealing with neutrosophic decision making was evaluated by Prof. Anjaan Mukherjee. Research of Prof. Mukherjee receives more than 700 citations for his works. Prof. Mukherjee is working with his group members with neutrosophic soft sets and its applications.

3.9 Dr.Pabitra Kumar Maji

Dr. Pabitra Kumar Maji, M. Sc., Post Doc., is an Assistant Professor of mathematics in Bidhan Chandra College, Asansol, West bengal. He works on soft set, fuzzy soft set, intuitionistic fuzzy set, fuzzy set, neutrosophic set, neutrosophic soft set, etc.,

Contribution:

In 2011, Maji [149] presented an application of neutrosophic soft set in object recognition problem based on multi-observer input data set. The author also introduced an algorithm to choose an appropriate object from a set of objects depending on some specified parameters.

In 2014, Maji, Broumi, and Smarandache [150] defined intuitionistic neutrosophic soft set over ring and proved some properties related to this concept. They also defined intersection, union, AND and OR operations over ring (INSSOR). Finally, the authors defined the product of two intuitionistic neutrosophic soft set over ring.

In 2015, Maji [151] presented weighted neutrosophic soft sets. The author presented an application of weighted neutrosophic soft sets in MADM problem. According "Googlescholar'', his publication includes 20 research paper having citations 5948.

Maji [152] studied the concept of weighted neutrosophic soft sets. The author considered a multi-observer decision-making problem as an application of weighted neutrosophic soft sets. We have considered here a recognition strategy based on multi-observer input parameter data set.

3.10 Dr. Harish Kumar Garg

Dr. Harish Garg is an Assistant Professor in the School of Mathematics, Thapar Institute of Engineering &Technology (Deemed University) Patiala. He completed his post graduation (M.Sc) in

Mathematics from Punjabi University Patiala, India in 2008 and Ph.D. from Department of Mathematics, Indian Institute of Technology (IIT) Roorkee, India in 2013. His research interest includes neutrosophic decision-making, aggregation operators, reliability theory, soft computing technique, fuzzy and intuitionistic fuzzy set theory, etc.

Contribution:

In 2016, Garg and Nancy [153] defined some operations of SVNNs such as sum, product, and scalar multiplication under Frank norm operations. The authors also defined some averaging and geometric aggregation operators and established their basic properties.The authors also established a decision-making strategy based on the proposed operators and presented an illustrative numerical example.

In 2017, Garg and Nancy [154] developed a nonlinear programming (NP) model based on TOPSIS to solve decision-making problems. At first, the authors-constructed a pair of the nonlinear fractional programming model based on the concept of closeness coefficient and then transformed it into the linear programming model.

Garg and Nancy [155] defined some new types of distance measures to overcome the shortcomings of the existing measures for SVNSs. The authors presented a comparison between the proposed and the existing measures in terms of counter-intuitive cases for showing validity. The authors also demonstrated the defined measures with hypothetical case studies of pattern recognition as well as medical diagnoses.

Garg and Nancy [156] studied the entropy measure of order α for single valued neutrosophic numbers. The authors established some desirable properties of entropy measure. The author also developed a MADM strategy based on entropy measures and solved a numerical example of investment problem.

Nancy and Garg [157] proposed an improved score function for ranking the single as well as interval-valued neutrosophic sets by incorporating the idea of hesitation degree between the truth and false degrees. The authors also presented an MADM strategy based on proposed function and solved a numerical example to show its practicality and effectiveness.

Garg and Nancy [158] introduced some new linguistic prioritized aggregation operators in the linguistic single-valued neutrosophic set (LSVNS) environment.The authors proposed some prioritized weighted and ordered weighted averaging as well as geometric aggregation operators for a collection of linguistic single-valued neutrosophic numbers and established their basic properties. The authors also proposed MADM strategy and solved a numerical example.

Dr. Garg research receives more than 2000 citations. Dr. Garg acts an active reviewer for reputed international journals and received certificate of outstanding in reviewing from "Computer & Industrial Engineering'', "Engineering Applications of Artificial Intelligence'', "Applied Soft Computing'', "Applied Mathematical Modeling'', etc. Dr. Garg acts as editor for many international journals.

3.11 Dr.Sukanto Bhattacharya

Sukanto Bhattacharya is a faculy member and associated with Deakin Business School, Deakin University.

Sukanto Bhattacharya [159] is the first researcher who employed utility theory to financial decision-making and obtained Ph. D. for applying neutrosophic probability in finance. His Ph. D. thesis covers a substantial mosaic of related concepts in utility theory as applied to financial decision-making. The author reviewed some of the classical notions of Benthamite utility and the normative utility paradigm. The author proposed some key theoretical constructs like the neutrosophicnotion of perceived risk and the entropic utility measure.
Khoshnevisan, and Bhattacharya [160] added a neutrosophic dimension to the problem of determining the conditional probability that a financial misrepresentation of the data set.
Prof. Bhattacharya is an active researcher and his works in neutrosophics are found in [159-163]. His research receives more than 380 citations.

4. Conclusions

We have presented a brief overview of the contributions of some selected Indian researchers who

conducted research in neutrosophic decision making. We briefly presented the contribution of the selected Indian neutrosophic researchers in MADM. In future, the contribution of Indian researchers such as W. B. V. Kandasamy, Pinaki Majumdar,Surapati Pramanik, Samarjit Kar, and other Indian mathematicians in developing neutrosophics can be studied. The study can also be extended for mathematicians from other countries who contributed in developing neutrosophic science. Decision making in neutrosophic hybrid environment is gaining much attention. So it is a promising field of research in different neutrosophic hybrid environment and the real cahllenge lies in the applications of the developed theories. Since some of the selected researchers are young, it is hoped that the researchers will do more creative works and new research regarding their contributions will have to be conducted in future.

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Single Valued Neutrosophic Hyperbolic Sine Similarity Measure Based MADM Strategy; Hybrid Binary Logarithm Similarity Measure for MAGDM Problems under SVNS Assessments; Generalizations of Neutrosophic Subalgebras in BCK/BCI-Algebras Based on Neutrosophic Points; Further results on (∈, ∈)-neutrosophic subalgebras and ideals in BCK/BCI-algebras; Commutative falling neutrosophic ideals in BCK-algebras; On Neutrosophic Soft Prime Ideal; Single Valued Neutrosophic Soft Approach to Rough Sets, Theory and Application; A novel approach to nano topology via neutrosophic sets; NC-VIKOR Based MAGDM Strategy under Neutrosophic Cubic Set Environment; Contributions of Selected Indian Researchers to Multi-Attribute Decision Making in Neutrosophic Environment: An Overview.

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